If $\frac{\partial u}{\partial t} = \nu \Delta u + f(\underset{\sim}{r})$

and $\frac{\partial u}{\partial t} = 0$ (i.e. time-stationary solutions)

get

$$-\Delta u = \frac{f(\underset{\sim}{r})}{\nu}$$ we get Poisson's Equation

if $f(\underset{\sim}{r}) = 0$ we get Laplace's Equation

$$-\Delta u = 0$$

This is the "nullspace solution of the operator $-\Delta$ (+ B.C.). The solution $u=0$ is the trivial but it also has nontrivial solutions.

<u>Rmk:</u> Can also get Laplace & Poisson Equations in other physical systems, e.g.

if $\underset{\sim}{E} = -\nabla u$

here u is the (scalar) electric potential, then $\underset{\sim}{E}$ is the electric field

$$-\nabla \cdot \nabla u = \frac{f(\underset{\sim}{r})}{\alpha}$$

f represents the charge distribution

or $$-\Delta u = \frac{f(\underset{\sim}{r})}{\alpha}$$

if $f=0 \Rightarrow -\Delta u = 0$.

In Fluid Mechanics:

If $\underset{\sim}{v} = -\nabla u$

then $-\nabla \cdot \nabla u = 0$

or $-\Delta u = 0$

is called "Stokes Flow"

& other applications are possible.

LAPLACE'S EQUATION
(elliptic PDE's)

let $V = V(\underset{\sim}{r})$ a scalar

$\underset{\sim}{r} = (x, y, z)$ for example, in 3D.

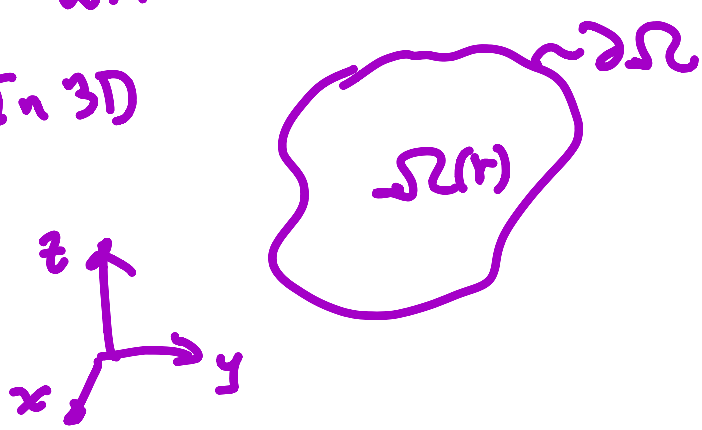
We will use separation of variables to find the solution to

$$\Delta V = 0$$

inside some domain $\Omega(\underset{\sim}{r})$

with B.C. on $\partial\Omega(\underset{\sim}{r})$

In 3D



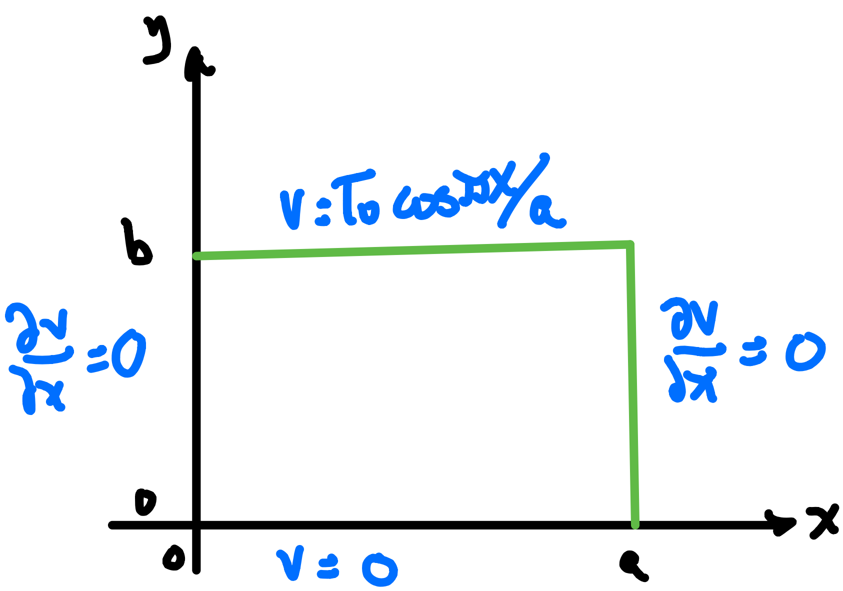
ex) in 2D: $\Delta V = 0$ $\quad 0 < x < a$, $0 < y < b$

$V = 0$ at $y = 0$

$V = T_0 \cos \frac{\pi x}{a}$ at $y = b$

$\frac{\partial V}{\partial x}(0, y) = 0$ at $x = 0$

$\frac{\partial V}{\partial x}(a, y) = 0$ at $x = a$



Cartesian geometry....

BVP

BVP
$$\begin{cases} \text{PDE} & \Delta v = 0 \\ \text{B.C} & \begin{cases} \frac{\partial v}{\partial x}(0,y) = \frac{\partial v}{\partial x}(a,y) = 0 \\ v(x,0) = 0 \\ v(x,b) = T_0 \cos\frac{\pi x}{a} \end{cases} \end{cases}$$

Try separation of variables

$$(\star) \quad v(x,y) = \varphi(x)\,\psi(y)$$

PDE: $\Delta v = \partial_{xx} v + \partial_{yy} v = 0$

subst $(\star)$ into PDE

$$\varphi_{xx}\psi + \varphi\psi_{yy} = 0$$

divide by $\varphi\psi$ to get

$$\frac{\varphi_{xx}}{\varphi} = -\frac{\psi_{yy}}{\psi} = -k^2$$

why this sign? because want SL in x:

or

(A) $\varphi_{xx} + k^2\varphi = 0$

(B) $\psi_{yy} - k^2\psi = 0$

Briefly: if we'd chosen $+k^2$ we would have gotten $\varphi_{xx} - k^2\varphi = 0$

$\psi_{yy} + k^2\psi = 0$

the solution of $\varphi(x) = Ae^{kx} + Be^{-kx}$

i.e. the hyperbolic functions

NOT S.L.

The solution of (A)

$$\varphi(x) = A\cos kx + B\sin kx$$

What B.C.?

$$\frac{\partial v}{\partial x}(0,y) = \frac{\partial v}{\partial x}(a,y) = 0$$

if $v = \varphi(x)\psi(y)$

$$\frac{\partial \varphi(0)}{\partial x}\psi(y) = \frac{\partial \varphi(a)}{\partial x}\psi(y) = 0$$

or $\varphi'(0) = \varphi'(a) = 0$ B.C.

to use on $\varphi_{xx} + k^2\varphi = 0$

$$\varphi' = -kA\sin kx + kB\cos kx$$

$\varphi'(0) = 0$ implies $B = 0$

$$\varphi'(a) = 0 = -kA\sin ka$$

$$k_n = \frac{n\pi}{a} \quad n = 1, 2 \ldots$$

$$\therefore \quad \varphi_n(x) = \cos\frac{n\pi x}{a} \quad n = 1, 2 \ldots$$

For (B) $\quad \psi_{yy} - k^2\psi = 0$

$$\psi(0) = 0$$

$$\psi(b) = T_0 \cos\frac{\pi x}{a}$$

$$\psi(y) = C\cosh k_n y + D\sinh k_n y$$

$$\psi(0) = 0 = C$$

$$\psi(y) = D\sinh k_n y$$

$$\therefore \quad v(x,y) = \sum_{n=1}^{\infty} a_n \sinh k_n y \cos\frac{n\pi x}{a}$$

Now we determine a_n:

$$v(x, y=b) = \sum_{n=1}^{\infty} a_n \sinh k_n b \cos\frac{n\pi x}{a}$$

$$= T_0 \cos\frac{\pi x}{a}$$ by B.C. at $y=0$

Using orthogonality: $n=1$ is the only term that survices

$$T_0 \cos\frac{\pi x}{a} = a_1 \sinh k_1 b \cos\frac{\pi x}{a}$$

$$\therefore \quad a_1 = \frac{T_0}{\sinh k_1 b}$$

$$\Rightarrow \quad V(x,y) = \frac{T_0}{\sinh \frac{\pi b}{a}} \sinh\frac{\pi y}{a} \cos\frac{\pi x}{a}$$