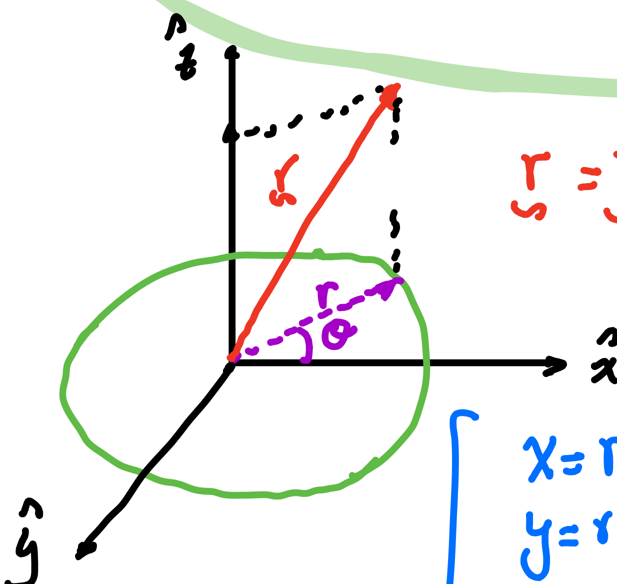
CYLINDRICAL/POLAR SYMMETRY

THE HEAT EQUATION & LAPLACE'S EQUATION

The Laplacian in 3D in cylindrical/polar coordinates is:

$$\Delta = \nabla \cdot \nabla = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_{\theta\theta} + \partial_{zz}$$



$$\underset{\sim}{r} = \underset{\sim}{r}(r, \theta, z)$$

$$\left[\begin{array}{ll} x = r\cos\theta & z = z \\ y = r\sin\theta & \\ \tan\theta = \frac{y}{x} & r^2 = x^2 + y^2 \end{array}\right]$$

In 2D (Polar):

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_{\theta\theta}$$

Rmk: The point here is that if you used separation of variables on these cylindrically-symmetric problems you do not obtain a SEPARATION OF VARIABLES solution. There are a variety of different coordinate systems that lead to separability (Depending on PDE and B.C.).

ex) Heat on a Disk (strictly radial symmetry)

let $u = u(r,t)$

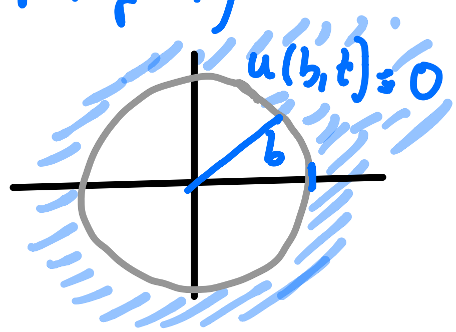
PDE $\quad \dfrac{\partial u}{\partial t} = \nu \Delta u = \nu \dfrac{1}{r}(r u_r)_r \qquad t>0,\ 0<r<b$

$$= \nu\left(u_{rr} + \frac{1}{r}u_r\right)$$

B.C. $\quad u(b,t) = 0$

(u is bounded)

$u(0,t)$ is bounded



I.C. $\quad u(r,0) = f(r) \qquad 0<r<b$

Separation of Variables Solution:

let $u = \psi(t)\,\varphi(r)$

Substitute into PDE + B.C.:

PDE: $\frac{1}{\nu}\psi_t\varphi = \psi\left(\varphi_{rr} + \frac{1}{r}\varphi_r\right)$ or

$$\frac{1}{\nu}\frac{\psi_t}{\psi} = \frac{\varphi_{rr} + \frac{1}{r}\varphi_r}{\varphi} = -k^2$$ separable!

(A) $\psi_t = -k^2\nu\psi$

(B) $\varphi_{rr} + \frac{1}{r}\varphi_r + k^2\varphi = 0$

$\varphi(r=b) = 0$

$\varphi(r=0)$ is bounded

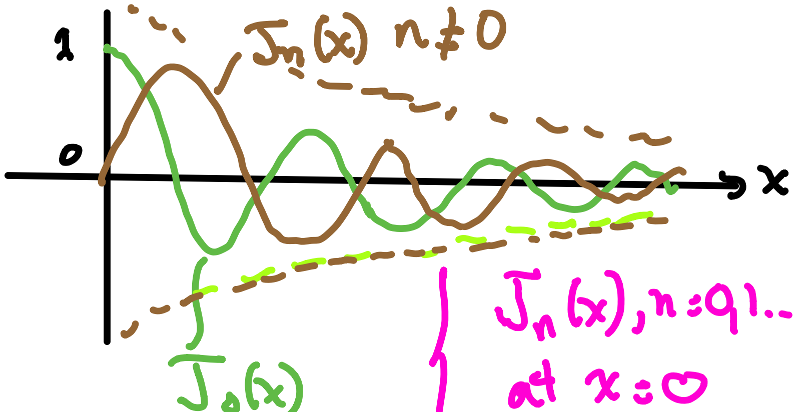
(A) has the familiar exponentially-decaying solution $\psi \sim \exp(-k^2\nu t)$

(B) Has the solution

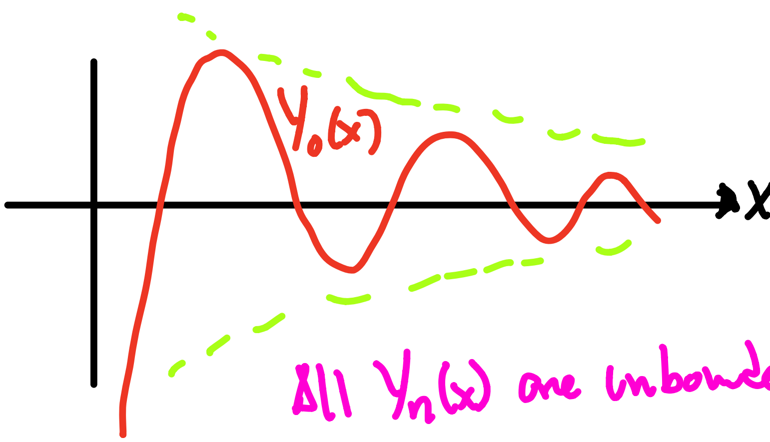
(*) $\varphi(r) = A\,J_0(kr) + B\,Y_0(kr)$

(B) is a S.L. problem with weight $w = r$.

Rmk: J_0 is Bessel function, order 0, "1st kind"
Y_0 is Bessel function, order 0, "2nd kind"



$J_n(x), n = 0, 1 \ldots$ are bounded at $x = 0$
$J_0(0) = 1$, $J_n(0) = 0$ $n \neq 0$



All $Y_n(x)$ are unbounded at $x = 0$

Since we require $u(r,t)$ bounded, including at $r = 0$, then $B = 0$

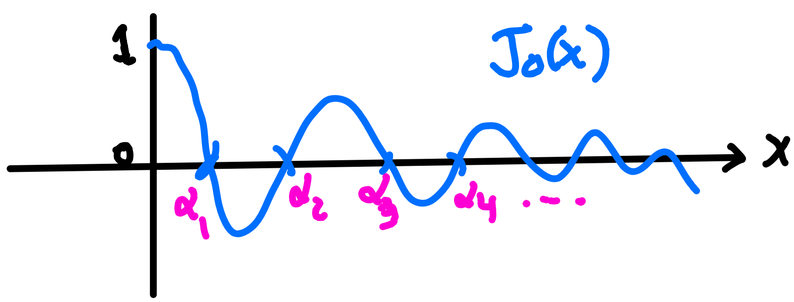
$\therefore \quad \varphi = A \, J_0(kr).$

Apply B.C. at $r = b$

$$\varphi(r=b) = A\, J_0(kb) = 0$$

$$k_n b = \alpha_n \qquad n = 1, 2 \ldots$$

α_n are the zeros of $J_0(x)$:



$$k_n = \frac{\alpha_n}{b} \qquad n = 1, 2 \ldots$$

$$\therefore\ u(r,t) = \sum_{n=1}^{\infty} a_n e^{-\nu k_n^2 t} J_0(k_n r)$$

Apply I.C.

$$(*) \qquad u(r, t=0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(k_n r)$$

To find a_n:

Multiply B.S. of (**) by $r\,J(k_m r)$ and integrate

$$\int_0^b r\, f(r)\, J_0(k_m r)\, dr$$

$$= \sum_{n=1}^{\infty} a_n \int_0^b J_0(k_n r)\, J_0(k_m r)\, r\, dr$$

Since $\{J_0(k_m r)\}_{m=1}^{\infty}$ are orthogonal wrt r:

$$a_m = \frac{1}{Z_m} \int_0^b r\, f(r)\, J_0(k_m r)\, dr \qquad m = 1, 2, \ldots$$

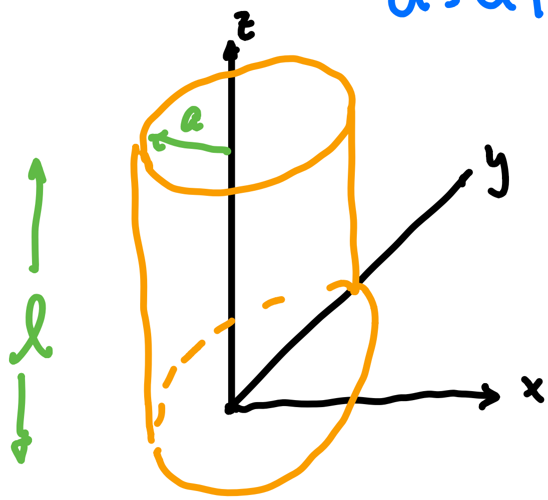
$$Z_m = \int_0^b r\, J_0^2(k_m r)\, dr, \text{ a number}$$

//

LAPLACE'S EQUATION WITH RADIAL and Z DEPENDENCE:

ex) Find Steady Temperature Distribution

$u = u(r, z)$



$\left\{ \begin{array}{l} u = f(z) = T_0 z(\ell - z) \\ \text{at } r = a \end{array} \right.$

u at the origin bounded

$u(r, \ell) = 0$

$u(r, 0) = 0$

$$\left\{ \begin{array}{ll} \text{PDE} & \Delta u = 0 = u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \\ \text{B.C.} & \left\{ \begin{array}{l} u(0, z) \text{ is bounded} \\ u(a, z) = T_0 z(\ell - z) \qquad (*) \\ u(r, 0) = u(r, \ell) = 0 \end{array} \right. \end{array} \right.$$

Separation of variables:

$u = H(z) R(r)$ and

substitute into PDE & B.C:

PDE $\quad \frac{1}{R}\left(R_{rr} + \frac{1}{r} R_r\right) = \frac{1}{H} H_{zz} = -\lambda^2$, a constant.

The choice of sign of the constant leads to SL in z:

that is,

$$\text{A} \quad \begin{cases} H_{zz} + \lambda^2 H = 0 \\ H(0) = H(\ell) = 0 \end{cases}$$

$$\text{B} \quad r^2 R_{rr} + r R_r + \lambda^2 r^2 R = 0$$

The solution to A $\quad H_n = \sin \frac{n\pi z}{\ell} \quad n = 1, 2 \ldots$

$$\therefore \lambda^2 = \left(\frac{n\pi}{\ell}\right)^2 \quad n = 1, 2 \ldots$$

The solution of B:

$$R = c_1 J_0\left(\frac{n\pi r}{\ell}\right) + c_2 Y_0\left(\frac{n\pi r}{\ell}\right)$$

since $u(r,z)$ is bounded $\Rightarrow c_2 = 0 \therefore$

$$u = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{\ell} J_0\left(\frac{n\pi r}{\ell}\right)$$

Apply B.C.

$$u(a,z) = T_0 z(l-z) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{l} J_0\left(\frac{n\pi a}{l}\right)$$

$$a_n = \frac{2}{Q_n l}\int_0^l T_0 z(l-z)\sin\frac{n\pi z}{l}\,dz$$

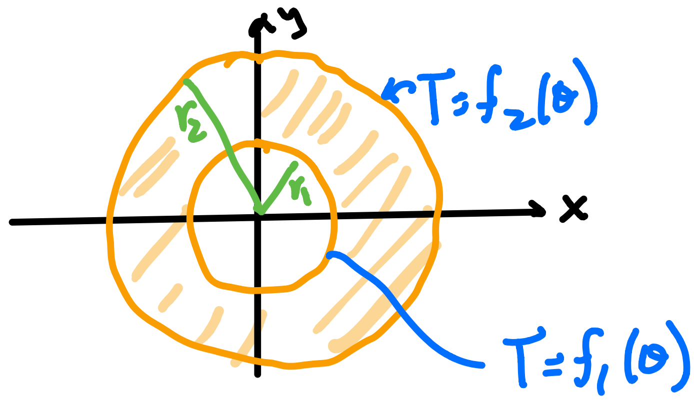
$$Q_n = J_0\left(\frac{n\pi a}{l}\right), \text{ a number.}$$

If we perform the integral, we get

$$a_n = \frac{4T_0 l^3}{Q_n l\, n^3\pi^3}\left[1-(-1)^n\right], \quad n=1,2\ldots$$

ex) Circular Annulus, Steady Temperature $T(r,\theta)$:

$$\begin{cases} \text{PDE} & \nabla^2 T = \dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \theta^2} = 0 \\ \text{B.C.} & T(r_1,\theta) = f_1(\theta) \quad T(r_2,\theta) = f_2(\theta) \end{cases}$$



$T = R(r)\, H(\theta)$ substitute into PDE & BC:

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} = 0$$

$$\frac{1}{R}\left(r^2R'' + rR'\right) = -\frac{H''}{H} = k^2$$ (choice gives SL in θ)

$$\therefore \begin{cases} r^2R'' + rR' - k^2R = 0 \\ H'' + k^2 H = 0 \end{cases}$$

Euler Equation (not Bessel)

The solution for the $R(r)$ equation

$$R_n = A_n r^n + B_n r^{-n} \qquad n \neq 0$$

$$R_0 = A_0 + B_0 \log r \qquad n = 0$$

$$H_n(\theta) = C_n \cos n\theta + B_n \sin n\theta$$

$$H_0(\theta) = C_0 + D_0 \theta$$

$$T(r,\theta) = (a_0 + b_0 \log r) \quad + (c_0 + d_0 \log r)\theta$$

$$+ \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

here $k = n$, an integer

(because the solution is periodic in θ)

(similarly, periodicity dictates that

$c_0 = d_0 = 0$)

$$\therefore T(r,\theta) = a_0 + b_0 \log r$$

$$+ \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos n\theta$$

$$+\sum_{n=1}^{\infty}\left(c_n r^n + d_n r^{-n}\right)\sin n\theta$$

Apply B.C.

$$f_{1,2}(\theta) = a_0 + b_0 \log r_{1,2}$$

$$+\sum_{n=1}^{\infty}\left(a_n r_{1,2}^n + b_n r_{1,2}^{-n}\right)\cos n\theta$$

$$+\sum_{n=1}^{\infty}\left(c_n r_{1,2}^n + d_n r_{1,2}^{-n}\right)\sin n\theta$$

$$\therefore \quad a_0 + b_0 \log r_{1,2} = \frac{1}{2\pi}\int_0^{2\pi} f_{1,2}(\theta)\, d\theta$$

$$a_n r_{1,2}^n + b_n r_{1,2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_{1,2}(\theta)\cos n\theta\, d\theta$$

$$c_n r_{1,2}^n + b_n r_{1,2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_{1,2}(\theta)\sin n\theta\, d\theta$$

//

Since:

$$r_1 = 0$$

$$r_2 = a$$

$\therefore b_0 = b_n = d_n = 0$

to avoid a singular solution.

In this case

$$\nabla^2 T = 0 \qquad r < a$$

For specificity, e.g.

$$T(a,\theta) \doteq f(\theta) = 3\cos 2\theta$$

$$\therefore T(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = 0$$ for our case

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta = 0 \text{ for our case}$$

$A_n = 0$ except $n = 2$ $A_n = 3$

$$T(r,\theta) = 3\left(\frac{r}{a}\right)^2 \cos 2\theta \text{ for our case.}$$

S'pse $r_2 \to \infty$ r_1 is finite

This is an infinite plate with a hole centered at $r = 0$.

To avoid unbounded solution

$$b_0 = a_n = c_n = 0 \qquad n = 1, 2 \ldots$$

$$\nabla^2 T = 0$$

$$T(r_1, \theta) = f(\theta)$$

$$T(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(\frac{r_1}{r}\right)^n (B_n \cos n\theta + D_n \sin n\theta)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

$$D_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

$f(\theta)$ is prescribed $\therefore$ all of the coefficients are known.