Discrete Convolution: to gain some intuition

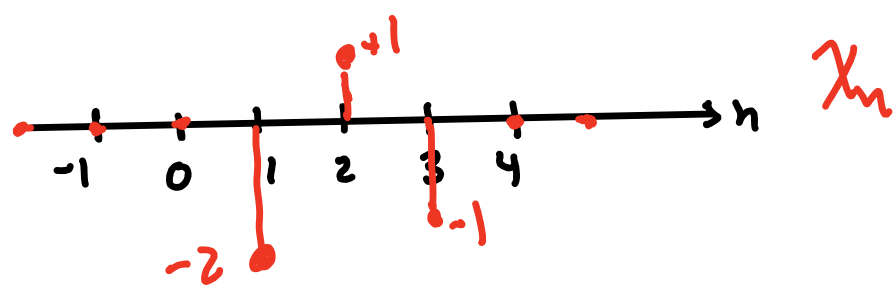
$$y_n = \sum_{j=-\infty}^{\infty} x_j h_{n-j} \quad n \in \mathbb{Z}$$

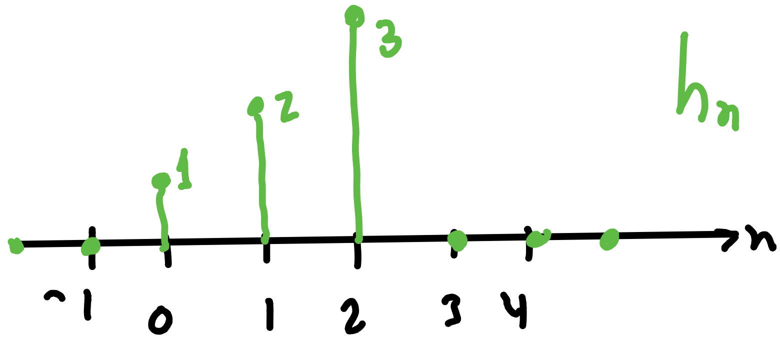
This is the definition of the discrete convolution of two sequences x_n, h_n $n \in \mathbb{Z}$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \quad t \in \mathbb{R}^1$$

is the continuous version, for the convolution of two functions $x(t), h(t), t \in \mathbb{R}^1$.

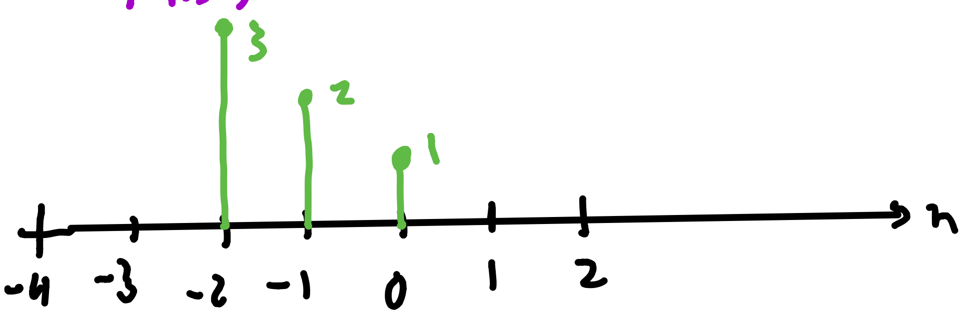
ex)



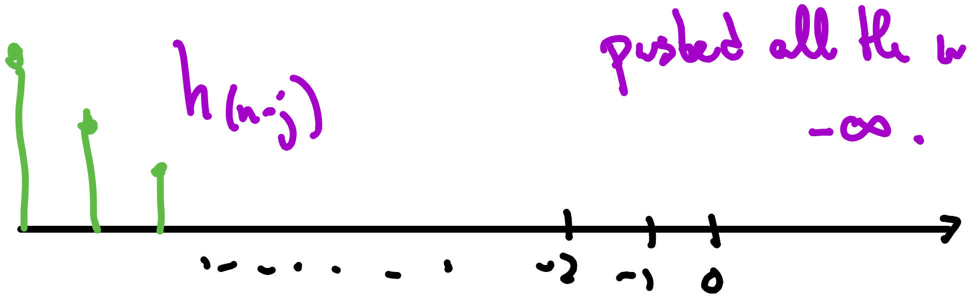


$$y_n = \sum_{j=-\infty}^{\infty} x_j h_{n-j} \quad n \in \mathbb{Z}$$

First, let's obtain h_{-n}



Next we let $h(n-j)$



pushed all the way to $-\infty$.

Next we slide h_{n-j} and compute

$$y_n = \sum_{j=-\infty}^{\infty} x_j h_{n-j}$$

We see that $y_n = 0$ for $n \leq 0$

for $n=1$ $\quad y_1 = \sum_{j=-\infty}^{\infty} x_j h_{1-j} = 1\cdot(-2) = -2$

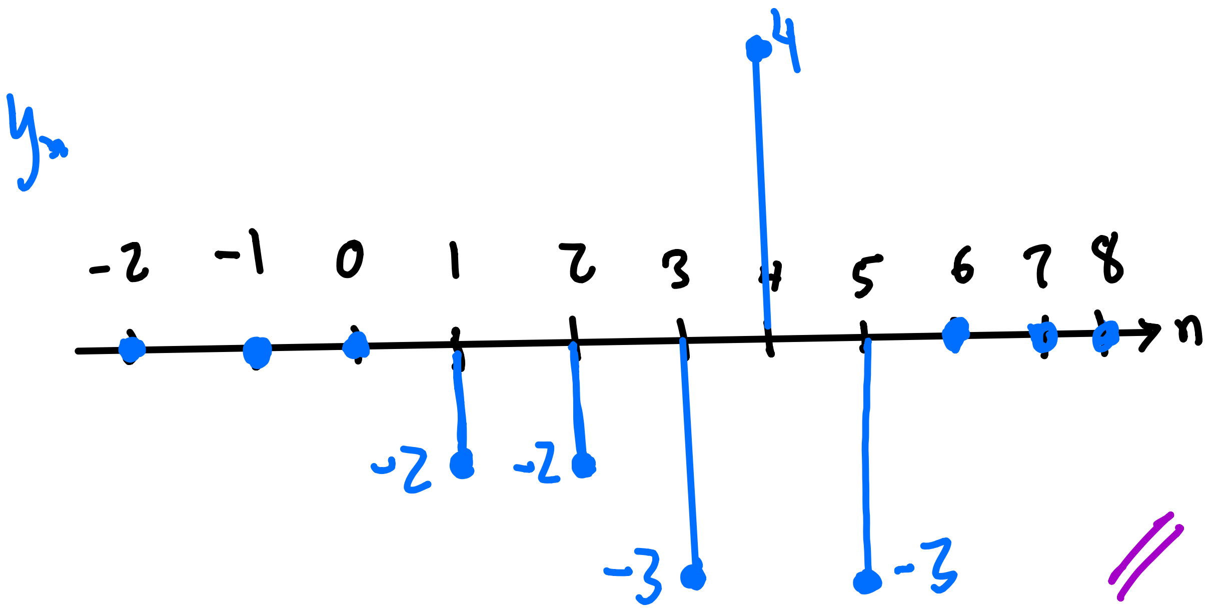
$n=2$ $\quad y_2 = \sum_{j=-\infty}^{\infty} x_j h_{2-j} = 1\cdot 2 + (2)(-2) = -2$

$n=3$ $\quad y_3 = \sum_{j=-\infty}^{\infty} x_j h_{3-j} = 3(-2) + 2\cdot 2 + (1)(-1) = -3$

$n=4$ $\quad y_4 = \sum_{j=-\infty}^{\infty} x_j h_{4-j} = 2(-1) + 3\cdot 2 = 4$

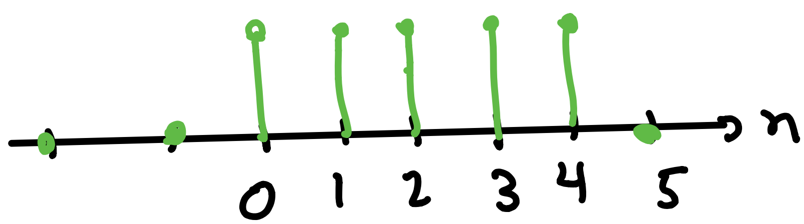
$n=5$ $\quad y_5 = 3\cdot(-1) = -3$

$n \geq 6$ $\quad y_n = 0$

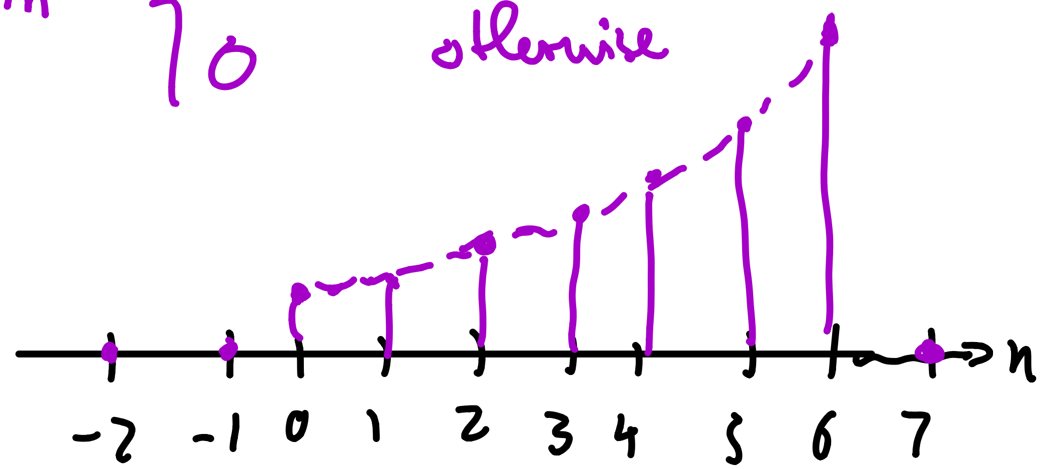


ex)

$$x_n = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$



$$h_n = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$



for $n < 0$ $\quad y_n = 0$

for $0 \le n \le 4$:

$$y_n = \sum_{j=-\infty}^{\infty} x_j h_{n-j} = \begin{cases} \alpha^{n-j} & 0 \le j \le n \\ 0 & \text{otherwise} \end{cases}$$

$$y_n = \sum_{j=0}^{n} \alpha^{n-j}, \text{ let } r = n-j$$

recall that, for $|x| \neq 1$ $\displaystyle\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$

$$y_n = \sum_{r=0}^{n} \alpha^r = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $n>4$ & $n-6 \leq 0$, i.e. $4<n\leq 6$,

$$x_j h_{n-j} = \begin{cases} \alpha^{n-j} & 0\leq j\leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$y_n = \sum_{j=0}^{4} \alpha^{n-j} = \alpha^n \sum_{j=0}^{4} \alpha^{-j} = \alpha^n \frac{1-\alpha^{-5}}{1-\alpha^{-1}}$$

$$= \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

for $n>6$ but $n-6\leq 4$, i.e, $6<n\leq 10$:

$$x_j h_{n-j} = \begin{cases} \alpha^{n-j} & (n-6)\leq j\leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$y_n = \sum_{j=n-6}^{4} \alpha^{n-j} \qquad \text{let } r = j-n+6$$

$$= \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-n} (\alpha^{-1})^r$$

$$= \alpha^6 \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} .$$

For $(n-6) < 4$ or $n > 4$

$$y_n = 0$$

In summary

$$
y_n = \begin{cases} 0 & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha} & 6 < n \le 10 \\ 0 & 10 < n \end{cases}
$$