What is a differential equation?

A differential equation is any equation containing one or more derivatives.

The simplest differential equation, therefore, is just a usual integration problem

$$y' = f(t).$$

Comment: The solution of the above is, of course, the indefinite integral of $f(t)$, $y = F(t) + C$, where $F(t)$ is any antiderivative of $f(t)$ and C is an arbitrary constant. Such a solution is called a *general solution* of the differential equation. It is a general form of a set of infinitely many functions, each differs from others by one (or more) constant term and/or constant coefficients, which all satisfy the differential equation in question. Every differential equation, if it does have a solution, always has infinitely many functions satisfying it. All of these solutions, differing from one another by one, or more, arbitrary constant / coefficient(s), are given by the formula of the general solution. Additional *auxiliary condition(s)*, which might appear as a problem demands, will be required to narrow down the solution set to one or a few specific functions from the formula of the general solution.

Classification of Differential Equations

Ordinary vs. partial differential equations

An *ordinary differential equation* (ODE) is a differential equation with a single independent variable, so the derivative(s) it contains are all ordinary derivatives.

A *partial differential equation* (PDE) is a differential equation with two or more independent variables, so the derivative(s) it contains are *partial derivatives*.

Order of a differential equation

The *order* of a differential equation is equal to the order of the highest derivative it contains.

Examples:

(1) $y' + y^5 = t^2 e^{-t}$ (first order ODE)

(2) $\cos(t)y' - \sin(t)\,y = 3t\cos(t)$ (first order ODE)

(3) $y'' - 3y' + 2y = e^{2t}\cos(5t)$ (second order ODE)

(4) $y^{(4)} + (y')^{30} = 0$ (fourth order ODE)

(5) $u_{xx} = 4u_{tt} + u_t$ (second order PDE)

(6) $y^{(5)} - (y''y') + 2y = 4e^{7t}$ (fifth order ODE)

Linear vs. nonlinear differential equations:

An n-th order ordinary differential equation is called *linear* if it can be written in the form:

$$\boxed{y^{(n)} = a_{n-1}(t)\, y^{(n-1)} + a_{n-2}(t)\, y^{(n-2)} + \ldots + a_1(t)\, y' + a_0(t)\, y + g(t).}$$

Where the functions a's and g are any functions of the independent variable, t in this instance. Note that the independent variable could appear in any shape or form in the equation, but the dependent variable, y, and its derivatives can only appear alone, in the first power, not in a denominator or inside another (transcendental) function. In other words, the right-hand side of the equation above must be a *linear function* of the dependent variable y and its derivatives. Otherwise, the equation said to be *nonlinear*.

In the examples above, (2) and (3) are linear equations, while (1), (4) and (6) are nonlinear. (5) is a linear partial differential equation, as each of the partial derivatives appears alone in the first power. The next example looks similar to (3), but it is a (second order) nonlinear equation, instead. Why?

(7) $\quad y'' - 3y' + 2y = e^{2t}\cos(5y)$

Exercises A-1.1:

1 – 10 Determine the order of each equation below. Also determine whether each is a linear or nonlinear equation.

1. $y' + t^3 y = \cos(t^2)$

2. $y''' + 11y'' - y' + e^{-6t} y = 2y \ln t$

3. $y'' = ty^5$

4. $5y' = t^5 y$

5. $y^3 + \sec(t) = y^{(6)} y'$

6. $y'' + 5y' + 4y = -e^{4t} y$

7. $(y')^8 - y = 1$

8. $y'' \cos(y) = t^3 \sin(t)\, y^{(5)}$

9. $e^t y^{(4)} + 3y'' - \cot(e^t)\, y = 2t^6 + y'$

10. $\dfrac{d^4}{dt^4}\left(t^2 y'\right) = \dfrac{ye^{8t}}{t}$

11. For what value(s) of n will the following equation be linear?

$$y' - 9y^n = t^{2n} \sin(3nt)$$

Answers A-1.1:

1. 1st order, linear; 2. 3rd order, linear; 3. 2nd order, nonlinear;
4. 1st order, linear; 5. 6th order, nonlinear; 6. 2nd order, linear;
7. 1st order, nonlinear; 8. 5th order, nonlinear; 9. 4th order, linear;
10. 5th order, linear; 11. When $n = 0$ or 1, the equation is linear.

Direction Field (a.k.a. Slope Field)

Direction field is a simple visualization tool that could be used to study the approximated behavior of the solutions of a first order differential equation

$$y' = f(t, y),$$

without having to solve it first.

What is it? First draw a grid on the ty-plane. Then for each point (t_0, y_0) on the grid compute the value $f(t_0, y_0)$. Note that $f(t_0, y_0) = y'$ is actually the instantaneous rate of change of a solution, $y = \varphi(t)$, of the given equation at the point (t_0, y_0). It, therefore, represents the slope of the line tangent to the solution whose curve passes through (t_0, y_0), at the exact point. Draw a short arrow at each such point (t_0, y_0) that is pointing in the direction given by the slope of the tangent line. After an arrow is drawn for every point of the grid, we can do "connecting-the-dots" and trace curves by connecting one arrow to the next arrow in the grid where the first is pointing at. Those curves traced this way are called *integral curves* (so called because, in effect, they each approximates an antiderivative of the function $f(t, y)$). Each integral curve approximates the behavior of a particular function that satisfies the given differential equation. The collection of all integral curves approximates the behaviors of the general solution of the equation.

Example: $y' = 2t$

What we are doing is constructing the graphs of some functions that satisfy the given differential equation by first approximating each solution function's local behavior at a point (t_0, y_0) using its linearization (i.e. the tangent line approximation). Then we obtain the longer-term behavior by connecting those local approximations, point-by-point moving among the grid, into curves that are fairly accurately resemble the actual graphs of those functions. We will look at this tool in more details in a later section, when we study **Autonomous Equations**.

Figure 1. The direction field of $y' = 2t$

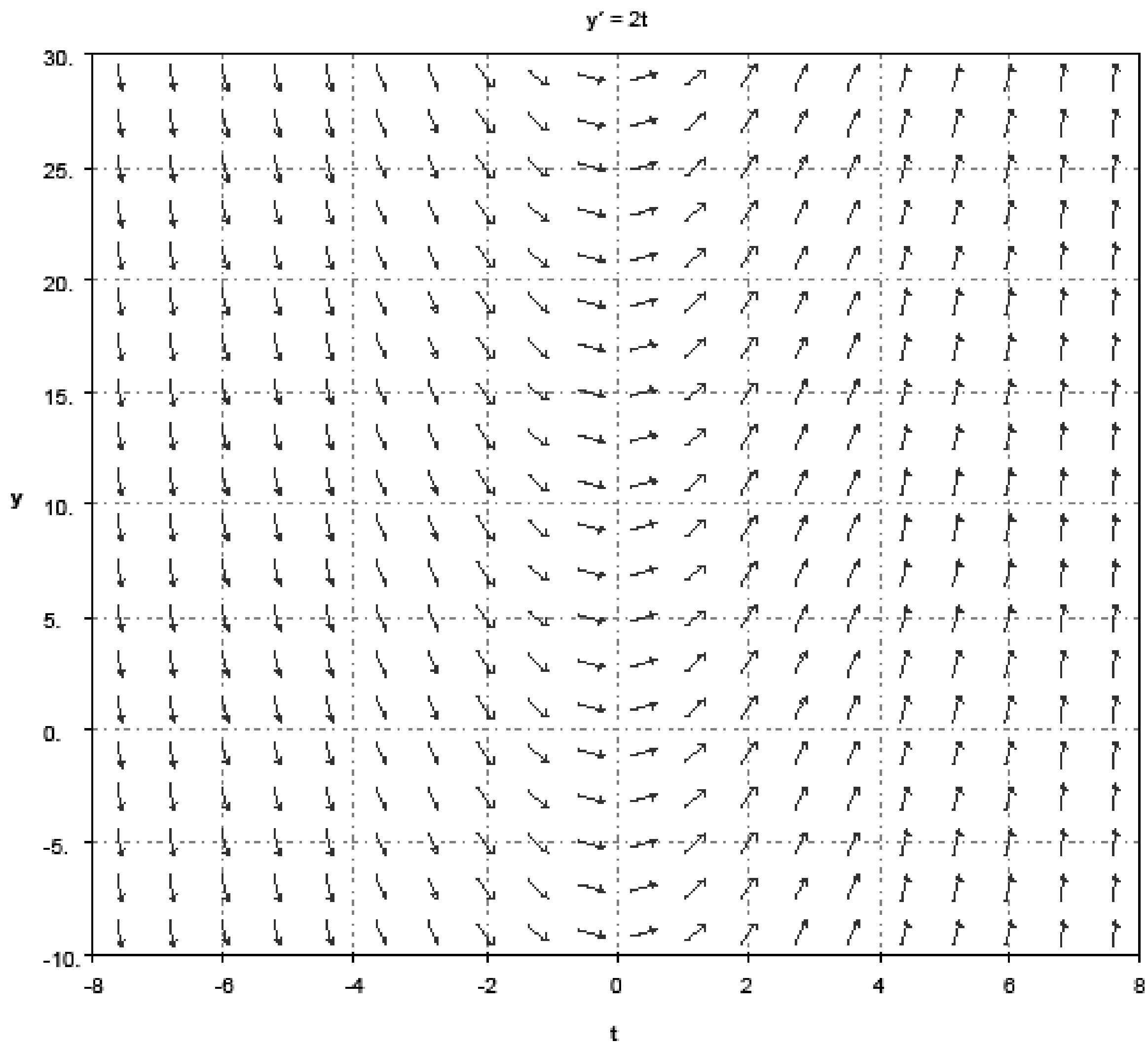
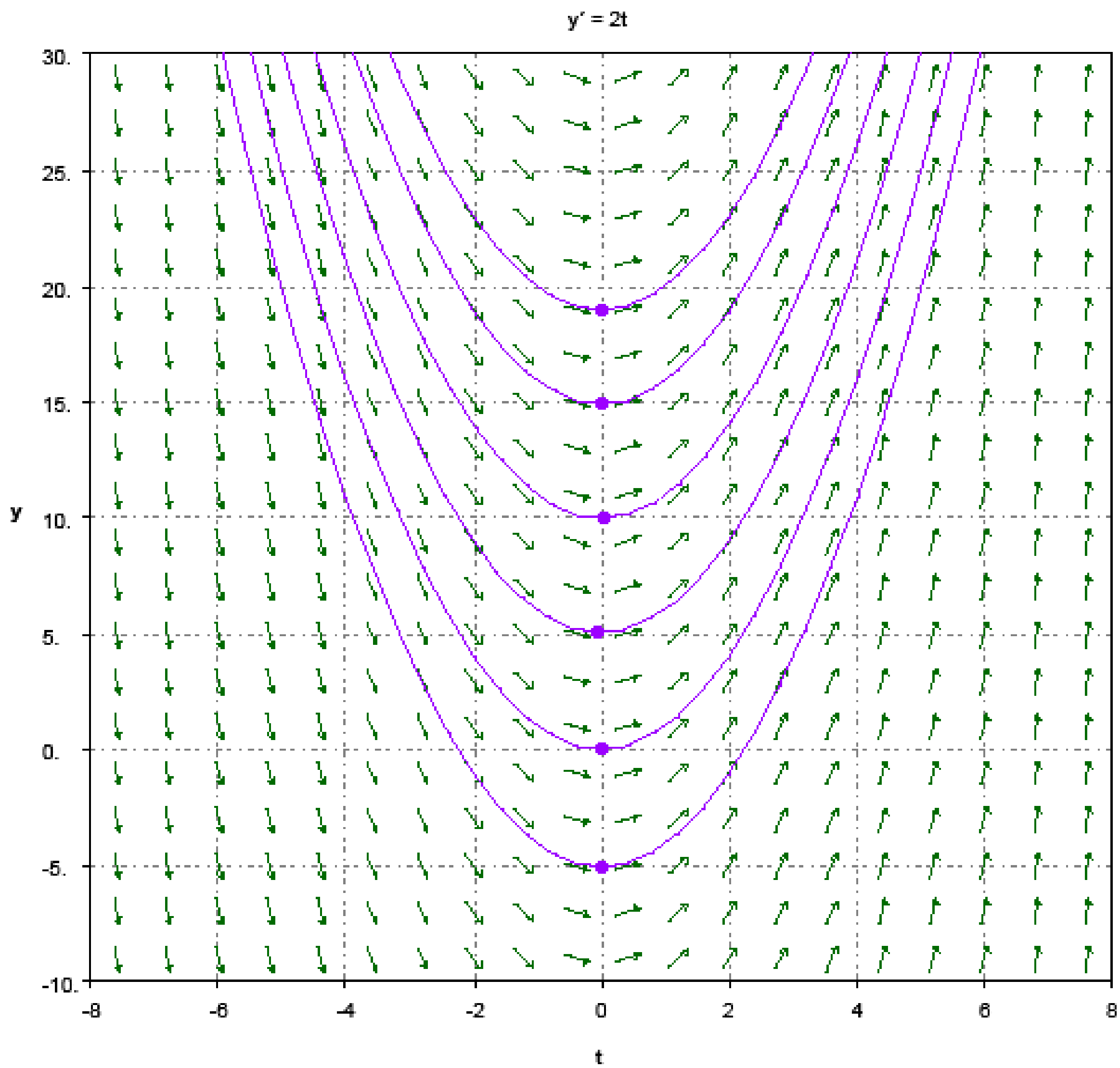


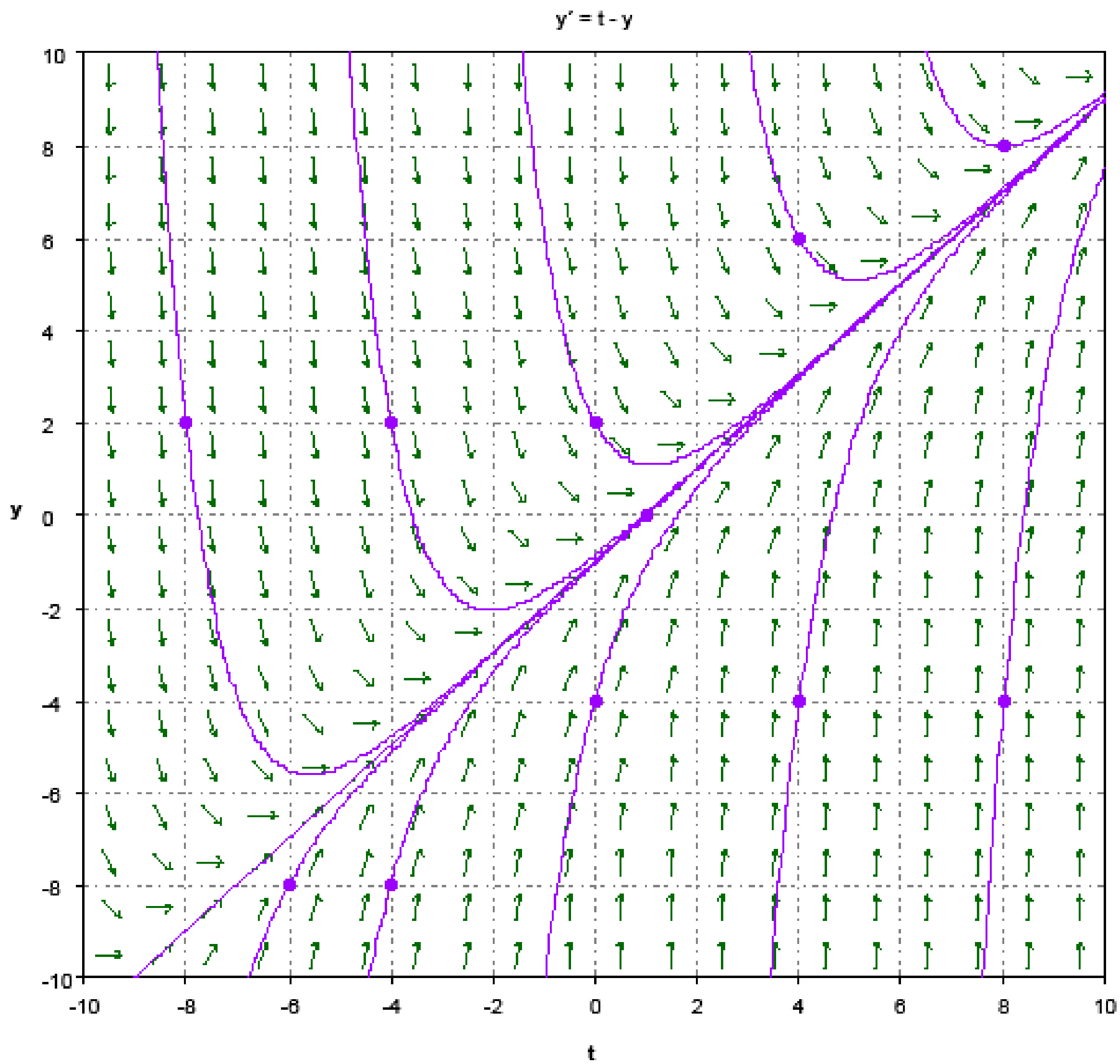
Figure 2. The direction field of $y' = 2t$ (with a few integral curves traced – approximating curves of the form $y = t^2 + C$).



Comment: Each integral curve, representing a specific function that satisfies the given differential equation – colloquially, such function is called a *particular solution* of the equation – is analogously a certain antiderivative. (In this present example, it is actually an antiderivative, that of $f(t, y) = 2t$.) The entire direction field shows the different behaviors of a collection of those particular solutions. In other words, it gives us a rough idea about the general solution of the differential equation. The direction field, in its entirety, is thus analogous to the indefinite integral of $f(t, y)$.

Next, let us examine a slightly more interesting direction field of another simple first order differential equation, $y' = t - y$. Even without knowing what its general solution is (yet), we can nevertheless readily deduce from its direction field the long-term behavior of its solutions, which all seem to behave like the line $y = t - 1$.

Figure 3. Another example: the direction field of $y' = t - y$



Comment: We will learn shortly how to solve this equation. The exact solutions are functions of the form $y = t - 1 + Ce^{-t}$. When $C = 0$, the solution is just the line $y = t - 1$, which appears as the slant asymptote of all other solutions in the above graph.

First Order Linear Differential Equations

A first order ordinary differential equation is *linear* if it can be written in the form

$$\boxed{y' + p(t)\, y = g(t)}$$

where p and g are arbitrary functions of t.

This is called the *standard form* or *canonical form* of the first order linear equation.

We'll start by attempting to solve a couple of very simple equations of such type.

Example: Find the general solution of the equation

$$y' - 2y = 0.$$

First let's rewrite the equation as $\dfrac{dy}{dt} = 2y$.

Then, assuming $y \neq 0$, divide both sides by y:

$$\frac{1}{y}\frac{dy}{dt} = 2$$

Multiply both sides by dt:

$$\frac{dy}{y} = 2dt$$

Now what we have here are two derivatives which are equal. It implies (as a consequence of the *Mean Value Theorem*) that the anti-derivatives of the two sides must differ only by a constant of integration. Integrate both sides:

$$\ln|y| = 2t + C$$

or, $$|y| = e^{(2t+C)} = e^{C}e^{2t} = C_1 e^{2t}$$

Where $C_1 = e^C$ is an arbitrary, but always positive constant.

To simplify one step farther, we can drop the absolute value sign and relax the restriction on C_1. C_1 can now be any positive or negative (but not zero) constant. Hence

$$y(t) = C_1 e^{2t}, \qquad C_1 \neq 0. \qquad (1)$$

Lastly, what happens if our eariler assumption that $y \neq 0$ is false? Well, if $y = 0$ (that is, when y is the constant function zero), then $y' = 0$ and the equation is reduced to

$$0 - 0 = 0$$

which is an expression that is always true. Therefore, the constant zero function is also a solution of the given equation. Not exactly by a coincident, it corresponds to the missing case of $C_1 = 0$ in (1).

As a result, the general solution is in the form

$$y(t) = Ce^{2t}, \qquad \text{for any constant } C.$$

That is, any function of this form, regardless of the value of C, will satisfy the equation $y' - 2y = 0$. While there are infinitely many such functions, no other type of functions could satisfy the equation.

The similar technique could also be used to solve this next example.

Example: For arbitrary constants r and k, $r \neq 0$, solve the equation

$$y' - r\,y = k.$$

We will proceed as before to rewrite the equation into equality of two derivatives. Then integrate both sides.

$$\frac{dy}{dt} = ry + k$$

Assuming $ry + k \neq 0$:

$$\frac{dy}{ry+k} = dt \qquad \rightarrow \qquad \int \frac{dy}{ry+k} = \int dt$$

Therefore, $$\frac{1}{r}\ln\left| ry + k \right| = t + C$$

Simplifying: $$\ln\left| ry + k \right| = rt + C_1 \quad \rightarrow \quad \left| ry + k \right| = e^{rt + C_1}$$

$\rightarrow \quad \left| ry + k \right| = e^{rt} e^{C_1}$, where e^{C_1} is an arbitrary positive constant.

Dropping the absolute value sign:

$ry + k = C_2\, e^{rt}$, $C_2 = \pm e^{C_1}$ is any nonzero constant.

That is, $y = \dfrac{1}{r}\left(C_2\, e^{rt} - k\right) = \dfrac{C_2}{r} e^{rt} - \dfrac{k}{r}$.

Lastly, it can be easily checked that if $ry + k = 0$, implying that y is the constant function $\frac{-k}{r}$, the given differential equation is again satisfied. This constant solution corresponds to the above general solution for the case $C_2 = 0$. Hence, the general solution now includes all possible values of the unknown arbitrary constant:

$$y = \frac{C}{r}e^{rt} - \frac{k}{r}, \qquad\qquad C \text{ is any constant.}$$

The Integrating Factor Method

In the previous examples of simple first order ODEs, we found the solutions by algebraically separate the dependent variable- and the independent variable- terms, and write the two sides of a given equation as derivatives, each with respect to one of the two variables. Then just integrate both sides and simplify to find the solution y. However, this process was feasible only because the equations in question were a special type, namely that they were both *separable*, in addition to being first order linear equations. They do, however, illustrated the main goal of solving a first order ODE, namely to use integration to removed the y'-term.

Most first order linear ordinary differential equations are, however, not separable. So the previous method will not work because we will be unable to rewrite the equation to equate two derivatives. In such instances, a more elaborate technique must be applied. How do we, then, integrate both sides?

Let's look again at the first order linear differential equation we are attempting to solve, in its standard form:

$$y' + p(t)\, y = g(t).$$

What we will do is to multiply the equation through by a suitably chosen function $\mu(t)$, such that the resulting equation

$$\mu(t)\, y' + \mu(t)p(t)\, y = \mu(t)g(t) \qquad (*)$$

would have integrate-able expressions on both sides. Such a function $\mu(t)$ is called an *integrating factor*.

Comment: The idea of integrating factor is not really new. Recall how you have integrated sec(x) in Math 141. The integral as given could not be integrated. However, after the integrand has been multiplied by a suitable from of 1, in this case (tan(x) + sec(x))/ (tan(x) + sec(x)), the integration could then proceed quite easily.

$$\int \sec x\,dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x}\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

Now back to the equation

$$\mu(t)\,y' + \mu(t)p(t)\,y = \mu(t)g(t) \qquad (*)$$

On the right side there is explicitly a function of t. So it could always, in theory at least, be integrated with respect to t. The left hand side is the more interesting part. Take another look of the left side of (*) and compare it with this following expression listed side-by-side:

$$\mu(t)\,y' + \mu(t)p(t)\,y \qquad \leftrightarrow \qquad \mu(t)\,y' + \mu'(t)\,y$$

The second expression is, by the product rule of differentiation, nothing more than $(\mu(t)\,y)'$. Notice the similarity between the two expressions. Suppose the simple differential equation $\mu(t)p(t) = \mu'(t)$ could be satisfied, we would then have

$$\mu(t)\,y' + \mu(t)p(t)\,y = \mu(t)\,y' + \mu'(t)\,y = (\mu(t)\,y)'$$

Trivially, then, the left side of (*) could be integrated with respect to t.

$$\int (\mu(t)\,y' + \mu(t)p(t)\,y)\,dt = \int (\mu(t)\,y)'\,dt = \mu(t)\,y$$

Hence, to solve (*) we integrate both sides:

$$\int (\mu(t)\, y' + \mu(t)p(t)\, y)\, dt = \int \mu(t)g(t)\, dt$$

$$\rightarrow \quad \mu(t)\, y = \int \mu(t)g(t)\, dt \qquad (**)$$

Therefore, the general solution is found after we divide the last equation through by the integrating factor $\mu(t)$.

But before we can solve for the general solution, we must take a step back and find this (almost magical!) integrating factor $\mu(t)$. We have seen on the last page that it must satisfies the equation $\mu(t)p(t) = \mu'(t)$. This is a simpler equation that can be solved by our first method of separate the variables then integrate:

$$p(t) = \frac{\mu'(t)}{\mu(t)}$$

$$\rightarrow \qquad \int p(t)\, dt = \ln|\mu(t)| + C$$

$$\rightarrow \qquad e^{\int p(t)dt} = e^{\ln|\mu(t)|}e^{C}$$

$$\rightarrow \qquad e^{\int p(t)dt} = C_1\mu(t)$$

This is the general solution, of course. We just need one instance of it. Since any nonzero function of the above form can be used as the integrating factor, we will just choose the simplest one, that of $C_1 = 1$. As a result

$$\boxed{\mu(t) = e^{\int p(t)\,dt}\,.}$$

Once it is found, we can immediately divide both sides of the equation (**) by $\mu(t)$ to find $y(t)$, using the formula

$$\boxed{y(t) = \frac{\int \mu(t) g(t)\,dt \;\;(+C)}{\mu(t)}}$$

Note: In order to use this integrating factor method, the equation must be put into the standard form first (i.e. y'-term must have coefficient 1). Else our formulas won't work.

Comment: As it turns out, what we have just discovered is a very powerful tool. <u>As long as we are able to integrate the two required integrals</u>, this integrating factor method can be used to solve any first order linear ordinary differential equation.

Example: We will use our new found general purpose method to again solve the equation

$$y' - r\,y = k, \qquad\qquad r \neq 0.$$

The equation is already in its standard form, with $p(t) = -\,r$ and $g(t) = k$.

The integrating factor is $\mu(t) = e^{\int -r\,dt} = e^{-r\,t}$.

The general solution is

$$y = \frac{1}{e^{-r\,t}}\left(\int e^{-r\,t} k\,dt\right) = e^{r\,t}\left(\frac{-k}{r}e^{-r\,t} + C\right) = \frac{-k}{r} + Ce^{r\,t}$$

That is it!

(It looks slightly different, but this is indeed the same solution we found a little earlier using a different method.)

Example: We have previously seen the direction field showing the approximated graph of the solutions of

$$y' = t - y.$$

Now let us apply the integrating factor method to solve it.

The equation has as its standard form,

$$y' + y = t.$$

Where $p(t) = 1$ and $g(t) = t$.

The integrating factor is $\mu(t) = e^{\int dt} = e^t$.

The general solution is, therefore,

$$y = \frac{1}{e^t}\left(\int te^t \, dt\right) = e^{-t}\left(te^t - \int e^t \, dt\right) = e^{-t}\left(te^t - e^t + C\right)$$

$$= t - 1 + Ce^{-t}.$$

Summary: Solving a first order linear differential equation

$$y' + p(t)\,y = g(t)$$

0. Make sure the equation is in the standard form above. If the leading coefficient is not 1, divide the equation through by the coefficient of y'-term first. (Remember to divide the right-hand side as well!)

1. Find the integrating factor:

$$\mu(t) = e^{\int p(t)dt}$$

2. Find the solution:

$$y(t) = \frac{\int \mu(t) g(t)\, dt \ \left(+C\right)}{\mu(t)}$$

This is the general solution of the given equation. Always remember to include the constant of integration, which is included in the formula above as "(+ C)" at the end. Like an indefinite integral (which gives us the solution in the first place), the general solution of a differential equation is a set of infinitely many functions containing one or more arbitrary constant(s).

Initial Value Problems (I.V.P.)

Every time we solve a differential equation, we get a general solution that is really a set of infinitely many functions that are all solutions of the given equation. In practice, however, we are usually more interested in finding some specific function that satisfies a given equation and also satisfies some additional behavioral requirement(s), rather than just finding an arbitrary function that is a solution. The behavioral requirements are usually given in the form of *initial conditions* that say the specific solution (and its derivatives) must take on certain given values (the *initial values*) at some prescribed initial time t_0. For a first order equation, the initial condition comes simply as an additional statement in the form $y(t_0) = y_0$. That is to say, once we have found the general solution, we will then proceed to substitute $t = t_0$ into $y(t)$ and find the constant C in the general solution such that $y(t_0) = y_0$. The result, if it could be found, is a specific function (or functions) that satisfies both the given differential equation, and the condition that the point (t_0, y_0) is contained on its graph. Such a problem where both an equation and one or more initial values are given is called an *initial value problem* (abbreviated as I.V.P. in the textbook). The specific solution thusly found is often called a *particular solution* of the differential equation.

Graphically, the general solution of a first order ordinary differential equation is represented by the collection of all integral curves in a direction field, while each particular solution is represented individually by one of the integral curves.

To summarize, an initial value problem consists of two parts:

1. A differential equation, and
2. A set of initial condition(s).

We first solve the equation to find the general solution (which contains one or more arbitrary constants or coefficients). Then we use the initial condition(s) to determine the exact value(s) of those constant(s). The result is a particular solution of the equation.

Example: Solve the initial value problem

$$t\,y' - 2y = t^3 e^t - 4, \qquad\qquad y(1) = 2.$$

First divide both sides by t.

$$y' - \frac{2}{t}y = t^2 e^t - \frac{4}{t}$$

$\rightarrow \qquad p(t) = -\frac{2}{t}$, and $\quad g(t) = t^2 e^t - \frac{4}{t}$.

The integrating factor is

$$\mu(t) = e^{\int \frac{-2}{t}dt} = e^{-2\ln|t|} = e^{\ln|t^{-2}|} = \left|t^{-2}\right| = t^{-2}.$$

The general solution is

$$y = \frac{1}{t^{-2}} \int t^{-2}\left(t^2 e^t - \frac{4}{t}\right)dt = t^2 \int \left(e^t - 4t^{-3}\right)dt = t^2\left(e^t + 2t^{-2} + C\right)$$

$$= t^2 e^t + 2 + Ct^2$$

Apply the initial condition

$$y(1) = 2 = 1^2 e^1 + 2 + C\,1^2 = e + 2 + C$$

$$0 = e + C \qquad\qquad \rightarrow \qquad\qquad C = -e$$

Therefore,

$$y = t^2 e^t + 2 - et^2.$$

Example: Solve the initial value problem

$$\cos(t)\,y' - \sin(t)\,y = 3t\cos(t), \qquad\qquad y(2\pi) = 0.$$

Divide through by $\cos(t)$: $\qquad y' - \tan(t)\,y = 3t$

$p(t) = -\tan(t)$ and $g(t) = 3t$

The integrating factor is $\mu(t) = e^{\int -\tan(t)\,dt}$. (What is this function?)

Use the u-substitution, let $u = \cos(t)$ then $du = -\sin(t)dt$:

$$\int -\tan(t)\,dt = \int \frac{-\sin(t)\,dt}{\cos(t)} = \int \frac{du}{u} = \ln|u| + C = \ln|\cos(t)| + C$$

Near $t_0 = 2\pi$, $\cos(t)$ is positive, so we could drop the absolute value.

Hence, $\mu(t) = e^{\int -\tan(t)\,dt} = e^{\ln(\cos(t))} = \cos(t)$.

$$y(t) = \frac{1}{\cos(t)}\int 3t\cos(t)\,dt = \frac{3}{\cos(t)}\left(t\sin(t) - \int \sin(t)\,dt\right)$$

$$= \frac{3}{\cos(t)}\left(t\sin(t) + \cos(t) + C\right) = 3t\tan(t) + 3 + C\sec(t)$$

$$y(2\pi) = 0 = 6\pi\tan(2\pi) + 3 + C\sec(2\pi) = 0 + 3 + C = 3 + C$$

$$C = -3$$

Therefore,

$$y(t) = 3t\tan(t) + 3 - 3\sec(t).$$

The Existence and Uniqueness Theorem (of the solution a first order linear equation initial value problem)

Does an initial value problem always a solution? How many solutions are there? The following theorem states a precise condition under which exactly one solution would always exist for a given initial value problem.

Theorem: If the functions p and g are continuous on the interval $\boldsymbol{I}$: $\alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \varphi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in $\boldsymbol{I}$, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

That is, the theorem guarantees that the given initial value problem will always have (*existence* of) exactly one (*uniqueness*) solution, on any interval containing t_0 as long as both $p(t)$ and $g(t)$ are continuous on the same interval. The largest of such intervals is called the *interval of validity* of the given initial value problem. In other words, the interval of validity is the largest interval such that (1) it contains t_0, and (2) it does not contain any discontinuity of $p(t)$ nor $g(t)$. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of either $p(t)$ or $g(t)$.

Note that, unless t_0 is actually a discontinuity of either $p(t)$ or $g(t)$, there always exists a non-empty interval of validity. If, however, t_0 is indeed a discontinuity of either $p(t)$ or $g(t)$, then the interval of validity will be empty. Clearly, in such a case the conditions that the interval must contain t_0 and that it must not contain a discontinuity of $p(t)$ or $g(t)$ will be contradicting.

If so, such an initial value problem is not guaranteed to have a unique solution at all.

Example: Consider the initial value problem solved earlier

$$\cos(t)y' - \sin(t)y = 3t\cos(t), \qquad y(2\pi) = 0.$$

The standard form of the equation is

$$y' - \tan(t)y = 3t$$

with $p(t) = -\tan(t)$ and $g(t) = 3t$. While $g(t)$ is always continuous, $p(t)$ has discontinuities at $t = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \pm7\pi/2, \ldots$ According to the Existence and Uniqueness Theorem, therefore, a continuous and differentiable solution of this initial value problem is guaranteed to exist uniquely on *any* interval containing $t_0 = 2\pi$ but not containing any of the discontinuities. The largest such intervals is $(3\pi/2, 5\pi/2)$. It is the interval of validity of this problem. Indeed, the actual solution $y(t) = 3t\tan(t) + 3 - 3\sec(t)$ is defined everywhere within this interval, but not at either of its endpoints.

How to find the interval of validity

For an initial value problem of a first order linear equation, the interval of validity, if exists, can be found using this following simple procedure.

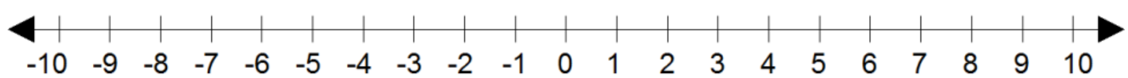
Given: $\qquad y' + p(t)y = g(t), \qquad y(t_0) = y_0.$

1. Draw the number line (which is the t-axis).

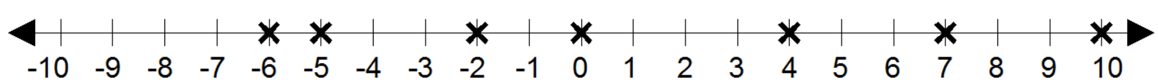
2. Find all the discontinuities of $p(t)$, and the discontinuities of $g(t)$. Mark them off on the number line.

3. Locate on the number line the initial time t_0. Look for the longest interval that contains t_0, but contains no discontinuities.

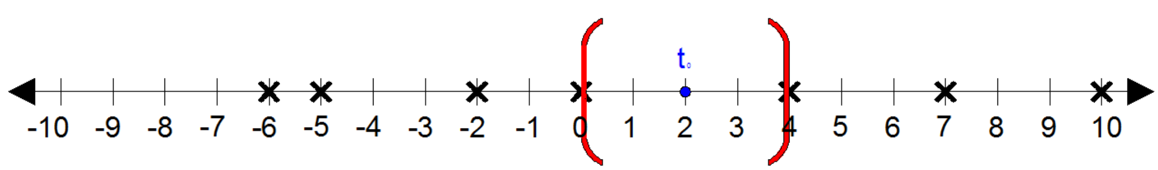
Step 1: Draw the t-axis.



Step 2: Mark off the discontinuities.



Step 3: Locate t_0 and determine the interval of validity.



Example: Consider the initial value problems

(a) $(t^2 - 81)y' + 5e^{3t}y = \sin(t)$, $\quad y(1) = 10\pi$

(b) $(t^2 - 81)y' + 5e^{3t}y = \sin(t)$, $\quad y(10\pi) = 1$

The equation is first order linear, so the theorem applies. The standard form of the equation is

$$y' + \frac{5e^{3t}}{t^2 - 81}y = \frac{\sin(t)}{t^2 - 81}$$

with $p(t) = \dfrac{5e^{3t}}{t^2 - 81}$ and $g(t) = \dfrac{\sin(t)}{t^2 - 81}$. Both have discontinuities at $t = \pm 9$. Hence, any interval such that a solution is guaranteed to exist uniquely must contain the initial time t_0 but not contain either of the points 9 and −9.

In (a), $t_0 = 1$, so the interval contains 1 but not ±9. The largest such interval is (−9, 9).

In (b), $t_0 = 10\pi$, so the interval contains 10π but neither of ±9. The largest such interval is $(9, \infty)$.

Remember that the value of y_0 does not matter at all, t_0 alone determines the interval.

Suppose the initial condition is $y(-100) = 5$ instead. Then the largest interval on which the initial value problem's solution is guaranteed to exist uniquely will be $(-\infty, -9)$.

Lastly, suppose the initial condition is $y(-9) = 88$. Then we would not be assured of a unique solution at all. Since $t = -9$ is both t_0 and a discontinuity of $p(t)$ and $g(t)$. The interval of validity would be, therefore, empty.

Depending on the problem, the interval of validity, if exists, could be as large as the entire real line, or arbitrarily small in length. The following example is an initial value problem that has a very short interval of validity for its unique solution.

Example: Consider the initial value problems

$$(t^2 - 10^{-2000000})y' + t\,y = 0, \qquad\qquad y(0) = \alpha.$$

With the standard form

$$y' - \frac{t}{t^2 - 10^{-2000000}}\,y = 0\,,$$

the discontinuities (of $p(t)$) are $t = \pm 10^{-1000000}$. The initial time is $t_0 = 0$. Therefore, the interval of validity for its solution is the interval $(-10^{-1000000}, 10^{-1000000})$, an interval of length 2×10^{-1000000} units!

However, the important thing is that somewhere on the t-axis a unique solution to this initial value problem exists. Different initial value α will give different particular solution. But the solution will each uniquely exist, at a minimum, on the interval $(-10^{-1000000}, 10^{-1000000})$.

Again, according to the theorem, the only time that a unique solution is not guaranteed to exist anywhere is whenever the initial time t_0 just happens to be a discontinuity of either $p(t)$ or $g(t)$.

Now suppose the initial condition is $y(0) = 0$. It should be fairly easy to see that the constant zero function $y(t) = 0$ is a solution of the initial value problem. It is of course the unique solution of this initial value problem. Notice that this solution exists for all values of t, not just inside the interval $(-10^{-1000000}, 10^{-1000000})$. It exists even at discontinuities of $p(t)$. This illustrates that, while outside of the interval of validity there is no guarantee that a solution would exist or be unique, the theorem nevertheless does not prevent a solution to exist, even uniquely, where the condition required by the theorem is not met.

Nonlinear Equations: Existence and Uniqueness of Solutions

A theorem analogous to the previous exists for general first order ODEs.

Theorem: Let the function f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$, containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \varphi(t)$ of the initial value problem

$$y' = f(t,y), \qquad y(t_0) = y_0.$$

This is a more general theorem than the previous that applies to all first order ODEs. It is also less precise. It does not specify a precise region that a given initial value problem would have a solution or that a solution, when it exists, is unique. Rather, it states a region that somewhere within there has to be part of it in which a unique solution of the initial value problem will exist. (It does not preclude that a second solution exists outside of it.)

The bottom line is that a nonlinear equation might have multiple solutions corresponding to the same initial condition. On the other hand it is also possible that it might not have a solution defined on parts of the region where f and $\partial f/\partial y$ are both continuous.

Example: Consider the (nonlinear) initial value problem

$$y' = t^2 y^{1/2}, \qquad y(0) = 0.$$

When $t = 0$, $\partial f/\partial y$ is not continuous. Therefore, it would not necessarily have a unique solution. Indeed, both $y = \dfrac{t^6}{36}$ and $y = 0$ are functions that satisfy the problem. (Verify this fact!)

Exercises A-1.2:

1 – 4 Find the general solution of each equation below.

1. $y' - t^2 y = 4t^2$

2. $y' + 10y = t^2$

3. $\dfrac{1}{t^2} y' - e^{t^3} y = 0$

4. $y' - y = 2e^t$

5 – 16 Solve each initial value problem. What is the largest interval in which a unique solution is guaranteed to exist?

5. $y' + 2y = te^{-t}$, $\quad y(0) = 2$

6. $y' - 11y = 4e^{6t}$, $\quad y(0) = 9$

7. $ty' - y = t^2 + t$, $\quad y(1) = 5$

8. $(t^2 + 1)y' - 2ty = t^3 + t$, $\quad y(0) = -4$

9. $y' + (2t - 6t^2)y = 0$, $\quad y(0) = -8$

10. $t^2 y' + 4ty = \dfrac{2}{t}$, $\quad y(-2) = 0$

11. $(t^2 - 49)y' + 4ty = 4t$, $\quad y(0) = 1/7$

12. $y' - y = t^2 + t$, $\quad y(0) = 3$

13. $y' + y = e^t$, $\quad y(0) = 1$

14. $ty' + 4y = 4$, $\quad y(-2) = 6$

15. $\tan(t)y' - \sec(t)\tan^2(t)y = 0$, $\quad y(0) = \pi$

16. $(t^2 + 1)y' + 2ty = 0$, $\quad y(3) = -1$

17 – 20 Without solving the initial value problem, what is the largest interval in which a unique solution is guaranteed to exist for each initial condition?

(a) $y(\pi) = 7$, (b) $y(1) = -9$, (c) $y(-4) = e$.

17. $(t+5)\,y' + \dfrac{(t-8)(t-1)}{t-3}\,y = \dfrac{t}{(t-6)(t+1)}$

18. $t^2 y' + \dfrac{t-2}{t+3}\,y = \sec(t/3)$

19. $(t^2 + 4t - 5)\,y' + \tan(2t)y = t^2 - 16$

20. $(4 - t^2)\,y' + \ln(6 - t)\,y = e^{-t}$

21. Find the general solution of $t^2 y' + 2ty = 2$. Then show that both the initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem?

Answers A-1.2:

1. $y = -4 + Ce^{t^3/3}$
2. $y = \dfrac{t^2}{10} - \dfrac{t}{50} + \dfrac{1}{500} + Ce^{-10t}$
3. $y = C\exp\left(\dfrac{1}{3}e^{t^3}\right)$
4. $y = 2te^t + Ce^t$
5. $y = te^{-t} - e^{-t} + 3e^{-2t}$, $(-\infty, \infty)$
6. $y = \dfrac{49}{5}e^{11t} - \dfrac{4}{5}e^{6t}$, $(-\infty, \infty)$
7. $y = t^2 + t\ln t + 4t$, $(0, \infty)$
8. $y = (t^2+1)(\ln(\sqrt{t^2+1}) - 4)$, $(-\infty, \infty)$
9. $y = -8\exp(2t^3 - t^2)$, $(-\infty, \infty)$
10. $y = \dfrac{1}{t^2} - \dfrac{4}{t^4}$, $(-\infty, 0)$
11. $y = \dfrac{t^4 - 98t^2 + 343}{(t^2-49)^2}$, $(-7, 7)$
12. $y = 6e^t - t^2 - 3t - 3$, $(-\infty, \infty)$
13. $y = \dfrac{1}{2}e^t + \dfrac{1}{2}e^{-t} = \cosh(t)$, $(-\infty, \infty)$
14. $y = 1 + 80t^{-4}$, $(-\infty, 0)$.
15. $y = \dfrac{\pi}{e}e^{\sec(t)} = \pi e^{\sec(t)-1}$, $(-\pi/2, \pi/2)$
16. $y = \dfrac{-10}{t^2+1}$, $(-\infty, \infty)$
17. (a) $(3, 6)$; (b) $(-1, 3)$; (c) $(-5, -1)$.
18. (a) $(0, 3\pi/2)$; (b) $(0, 3\pi/2)$; (c) $(-3\pi/2, -3)$.
19. (a) $(3\pi/4, 5\pi/4)$; (b) no such interval exists; (c) $(-5, -5\pi/4)$.
20. (a) $(2, 6)$; (b) $(-2, 2)$; (c) $(-\infty, -2)$.
21. $y = \dfrac{2t + C}{t^2}$; they both have $y = \dfrac{2t-1}{t^2}$ as the solution; no, different initial conditions could nevertheless give the same unique solution.

Separable Differential Equations

A first order differential equation is *separable* if it can be written in the form

$$\boxed{M(x) + N(y)\,y' = 0,}$$

where $M(x)$ is a function of the independent variable x only, and $N(y)$ is a function of the dependent variable y only. It is called separable because the independent and dependent variables could be moved to separate sides of the equation:

$$N(y)\frac{dy}{dx} = -M(x)\,.$$

Multiplying through by dx,

$$N(y)dy = -M(x)dx.$$

A general solution of the equation can then be found by simply integrating both sides with respect to each respective variable:

$$\boxed{\int N(y)dy = -\int M(x)dx + C\,.}$$

This is the *implicit* general solution of the equation, where y is defined implicitly as a function of x by the above equation relating the antiderivatives, with respect to their individual variables, of $M(x)$ and $N(y)$.

An *explicit* general solution, in the form of $y = f(x)$, where y is explicitly defined by a function $f(x)$ which itself satisfies the original differential equation, could be found (in theory, although not always in practice) by simplifying the implicit solution and solve for y.

Example: Solve $e^y \dfrac{dy}{dx} - x - x^3 = 0$

First, separate the *x*- and *y*-terms.

$$e^y \frac{dy}{dx} = x^3 + x$$

Then multiply both sides by *dx* and integrate

$$\int e^y \, dy = \int \left(x^3 + x\right) dx$$

$$e^y = \frac{x^4}{4} + \frac{x^2}{2} + C \qquad \text{(implicit solution)}$$

or,

$$y = \ln\left(\frac{x^4}{4} + \frac{x^2}{2} + C\right) \qquad \text{(explicit solution)}$$

Suppose there is, in addition, an initial condition of $y(1) = 2$. We can solve for the constant *C* by applying this initial condition:

$$e^2 = \frac{1^4}{4} + \frac{1^2}{2} + C \qquad \rightarrow \qquad C = e^2 - \frac{3}{4}$$

Finally,

$$e^y = \frac{x^4}{4} + \frac{x^2}{2} + e^2 - \frac{3}{4} \qquad \text{(implicit solution)}$$

or,

$$y = \ln\left(\frac{x^4}{4} + \frac{x^2}{2} + e^2 - \frac{3}{4}\right) \qquad \text{(explicit solution)}$$

Example: Solve the initial value problem

$$y' = \sqrt{1 - y^2}\cos(t), \qquad\qquad y(0) = 0.$$

Separate the variables and integrate, we have

$$\frac{dy}{\sqrt{1 - y^2}} = \cos(t)\,dt$$

$$\int \frac{dy}{\sqrt{1 - y^2}} = \int \cos(t)\,dt$$

$$\arcsin(y) = \sin(t) + C.$$

Apply the initial condition to solve for C, the (implicit) particular solution is

$$\arcsin(0) = \sin(0) + C$$

$$0 = 0 + C \qquad \rightarrow \qquad C = 0$$

$$\arcsin(y) = \sin(t).$$

The explicit particular solution can be found easily:

$$y = \sin(\sin(t)).$$

Question: How would the solution differ if the initial condition is $y(0) = 1$? (What happens when $y_0 = 1$?)

Example: Solve the initial value problem

$$y' = \frac{3t^2 + 4t - 5}{2y - 10}, \qquad y(1) = -2.$$

$$(2y - 10)\, y' = 3t^2 + 4t - 5$$

$$(2y - 10)\, dy = (3t^2 + 4t - 5)\, dt$$

$$\int (2y - 10)\, dy = \int (3t^2 + 4t - 5)\, dt$$

(Implicit) general solution is $y^2 - 10y = t^3 + 2t^2 - 5t + C$.

The initial condition says that when $t = 1$, $y = -2$, so substitute those two values into the general solution:

$$(-2)^2 - 10(-2) = 1^3 + 2(1)^2 - 5 + C$$

$$24 = -2 + C \qquad \rightarrow \qquad C = 26$$

The (implicit) particular solution is $y^2 - 10y = t^3 + 2t^2 - 5t + 26$.

What is the explicit solution? We will solve explicitly for y by first using completing-the-square to simplify the left side:

$$y^2 - 10y + 25 = t^3 + 2t^2 - 5t + 26 + 25$$

$$(y - 5)^2 = t^3 + 2t^2 - 5t + 51$$

$$y - 5 = \pm\sqrt{t^3 + 2t^2 - 5t + 51}$$

$$y(t) = 5 \pm \sqrt{t^3 + 2t^2 - 5t + 51} \qquad \text{(Which one?)}$$

It is necessary to determine which one of the two expressions above is the actual solution of this problem. Both expressions are derived from the same implicit solution of the given equation. Therefore, they would both satisfy the equation. However, there is a unique solution to this initial value problem, as we know. We do have a clue regarding the true identity of the solution. The clue is in the form of the initial condition, $y(1) = -2$. Let us check. Apply the initial condition to both expressions:

$$-2 = y(1) = 5 \pm \sqrt{1^3 + 2(1)^2 - 5 + 51} = 5 \pm \sqrt{49}\,.$$

Since $-2 = 5 - \sqrt{49}$, the correct explicit solution must be the expression with the minus sign.

$$y(t) = 5 - \sqrt{t^3 + 2t^2 - 5t + 51}\,.$$

Summary: Solving a separable differential equation

$$M(x) + N(y)\,y' = 0$$

1. Rearrange the equation into the form below, separating it into a dependent variable part and an independent variable part:

$$N(y)\,y' = -\,M(x).$$

Then convert both sides into derivatives by multiplying through with *dx*.

$$N(y)\,dy = -\,M(x)\,dx.$$

2. Integrating both sides to find the implicit general solution:

$$\int N(y)dy = -\int M(x)dx + C$$

The constants of integration should be combined and put into only one side (by convention, the independent variable side) of the equation.

3. If necessary / feasible, an explicit general solution, $y = f(x)$ can be found by simplifying the implicit solution and solve for *y*.

Exercises A-1.3:

1 – 6 Find the general solution of each system below.

1. $y' = t^2 y^{1/2}$

2. $y' = \dfrac{y}{t}$

3. $y' = \dfrac{\sin t}{\cos y}$

4. $y' = \dfrac{\sec y}{x^2 + 4}$

5. $y' = y^3 + 4y$

6. $y' = \cot y$

7 – 17 Solve the following initial value problem.

7. $y' = \sqrt{1 - y^2}\cos(t)$, $y(0) = 1$

8. $y' = \dfrac{y}{x^2 - 1}$, $y(2) = 1$

9. $y' = \dfrac{1}{xy^2}$, $y(1) = 3$

10. $y' - t^2 y = 4t^2$, $y(1) = 2$

11. $y' = 6t^2 y - 2ty$, $y(0) = -8$

12. $y' - y^2 = 4$, $y(0) = -2$

13. $y' = (1 + y^2)\sec^2 x$, $y(0) = -1$

14. $y' = e^{2y}\, t\sin(4t)$, $y(\pi) = -1$

15. $y' = \dfrac{2x^3 - 4x + 1}{y + 2}$, (a) $y(0) = 1$, (b) $y(1) = -4$.

16. $y\,y' = 2t(y^2 + 5)$, (a) $y(0) = 2$, (b) $y(-1) = -4$.

17. $y' = \dfrac{xy}{\ln(y)}$, (a) $y(4) = 1$, (b) $y(-2) = e$.

18. $y' - y = y^2$, (a) $y(0) = -2$, (b) $y(-2) = 0$.

Answers A-1.3:

2. $y = Ct$
3. $y = \sin^{-1}(-\cos t + C)$
4. $y = \sin^{-1}\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C\right)$
5. $\frac{1}{8}\ln\left(\frac{y^2}{y^2+4}\right) = t + C$ (in implicit form), and $y = 0$.
6. $y = \cos^{-1}(e^{-t+C})$
7. $y = 1$
8. $y = \sqrt{3\left|\frac{x-1}{x+1}\right|}$
9. $y = \sqrt[3]{3\ln|x| + 27}$
10. $y = 6e^{-1/3}e^{t^3/3} - 4$
11. $y = -8\exp(2t^3 - t^2)$
12. $y = 2\tan(2x - \pi/4)$
13. $y = \tan(\tan(x) - \pi/4)$
14. $y = \frac{-1}{2}\ln\left(\frac{1}{2}t\cos(4t) - \frac{1}{8}\sin(4t) + e^2 - \frac{\pi}{2}\right)$
15. (a) $y = -2 + \sqrt{x^4 - 4x^2 + 2x + 9}$, (b) $y = -2 - \sqrt{x^4 - 4x^2 + 2x + 5}$
16. (a) $y = \sqrt{9\exp(2t^2) - 5}$, (b) $y = -\sqrt{21\exp(2t^2 - 2) - 5}$
17. (a) $\ln^2(y) = \frac{x^2}{2} - 8$, (b) $\ln^2(y) = \frac{x^2}{2} - 1$ (in implicit form).
18. (a) $\ln\left|\frac{y}{y+1}\right| = t + \ln(2)$ (in implicit form), (b) $y = 0$

Applications of First Order Equations

I. Mixing solution

A mixing tank initially contains Q_0 amount salt (solute) dissolved in S_0 amount of water (solvent). Additional salt water (solution) of concentration c_i flows into the tank at a rate r_i. Assume the content of the mixing tank is stirred very rapidly such that the solution within is always of uniform concentration. The mixed content is then pumped out of the tank for use elsewhere at a rate r_o. Find the amount (mass) or concentration (mass/volume) of salt contained in the tank at any time $t > 0$.

Denote: $Q(t)$ = amount of solute in tank at time t
$S(t)$ = volume of solution in tank at time t

An expression for $S(t)$ is simple to derive: Since there are initially S_0 amount in the tank; and during each unit of time r_i amount flows into and r_o amount flows out of the tank, for a net change of $(r_i - r_o)$ per unit time. Therefore,

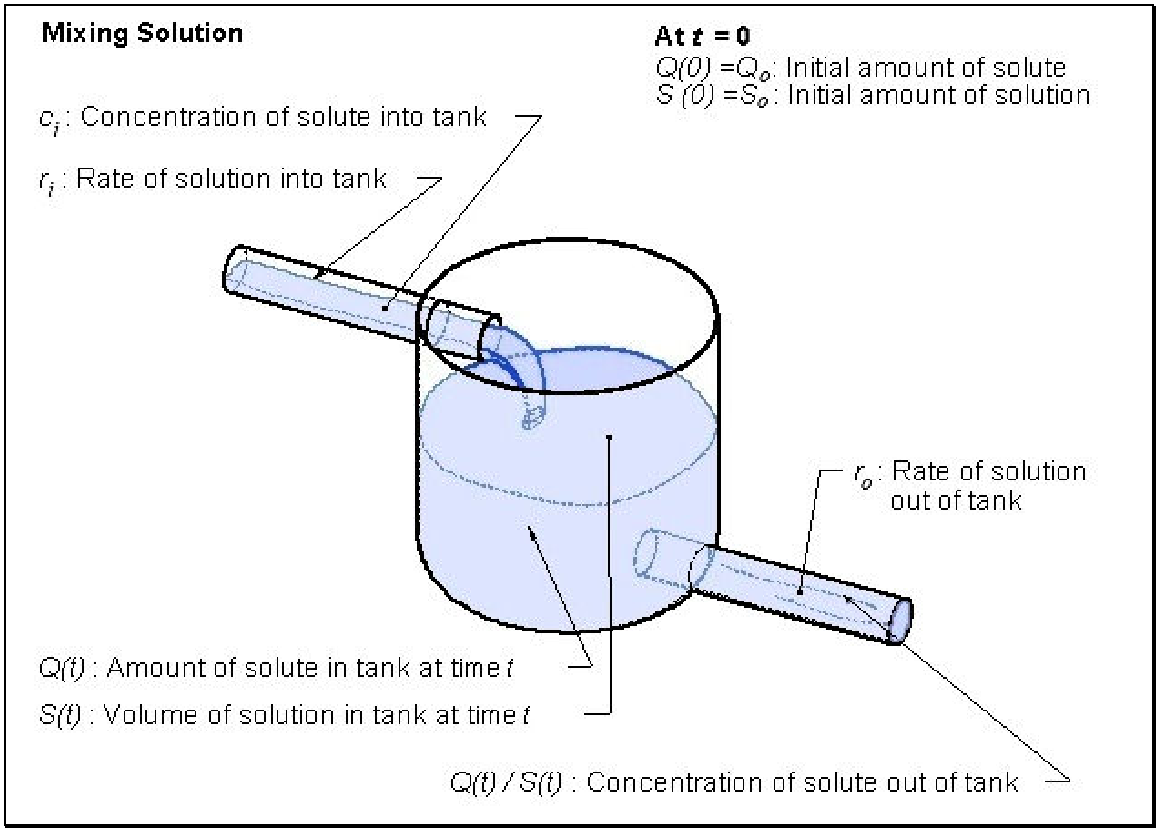
$$\boxed{S(t) = S_0 + (r_i - r_o)\,t}$$

Next, we want to come up with an equation that governs Q. The general form of the differential equation that governs the amount of solute in the mixing tank, $Q(t)$, at any time $t > 0$ is:

$$\boxed{Q' = (\text{rate of solute flowing in}) - (\text{rate of solute flowing out})}$$

The rate of solute in/out is equal to

$$\boxed{(\text{rate of solution in/out}) \times (\text{concentration of solution in/out})}$$



Therefore, the necessary initial value problem is

$$\boxed{Q' = r_i\, c_i - r_o \frac{Q}{S(t)}, \qquad Q(0) = Q_0}$$

The equation is a first order linear equation with the standard form

$$Q' + \frac{r_0}{S(t)} Q = r_i\, c_{i.}.$$

Consequently, <u>it can always be solved using the integrating factor method</u> we have already seen.

The constant-volume mixing problem (w/ constant rates, $r_i = r_o = r$)

In this case, $S(t) = S_0$. The mixing problem becomes

$$Q' + \frac{r}{S_0} Q = r c_i, \qquad Q(0) = Q_0$$

First identify $\qquad p(t) = \frac{r}{S_o}$, and $\qquad g(t) = r c_i$.

The integrating factor is $\mu(t) = e^{\int \frac{r}{S_0} dt} = e^{\frac{r}{S_0} t}$

$$Q(t) = \frac{1}{e^{rt/S_0}} \left(\int r c_i e^{\frac{r}{S_0} t} dt \right) = e^{\frac{-r}{S_0} t} \left(\frac{r c_i S_0}{r} e^{\frac{r}{S_0} t} + C \right) = c_i S_0 + C e^{\frac{-r}{S_0} t}$$

$$Q(0) = Q_0 = c_i S_0 + C e^0 = c_i S_0 + C \qquad \rightarrow \qquad C = Q_0 - c_i S_0$$

Therefore,

$$Q(t) = c_i S_0 + (Q_0 - c_i S_0) e^{\frac{-r}{S_0} t}.$$

The concentration as a function of time is $Q(t) / S(t) = Q(t) / S_0$.

The *limiting concentration* is $\lim_{t \to \infty} \frac{Q(t)}{S(t)} = \frac{c_i S_0}{S_0} = c_i$. That is, after a very long time, the concentration of the content of the tank will approach the concentration of the new inflow. (Since eventually every last drop of the original content will be flushed out of the tank and be replaced by the inflow solution.)

Example: A swimming pool initially contains 1000 m^3 of stale, un-chlorinated water. Water containing 2 grams per m^3 of chlorine flows into the pool at a rate of 4 m^3 per minute. The well-mixed content of the pool is drained at the same rate. Find the time when the chlorine concentration in the pool reaches 1 gram per m^3.

In this problem, $r_i = r_o = 4$. Therefore, it is a constant volume problem, and the initial volume $S_0 = 1000 = S(t)$. There is initially no solute (chlorine, in this case) in the pool, hence $Q(0) = 0$. The inflow concentration $c_i = 2$. We can then set up the following initial value problem

$$Q' = (4)(2) - \frac{4}{1000}\,Q = 8 - \frac{1}{250}\,Q \quad , \qquad Q(0) = 0.$$

In standard form: $$Q' + \frac{1}{250}\,Q = 8\,, \qquad Q(0) = 0.$$

Where $p(t) = \dfrac{1}{250}$, and $g(t) = 8$.

The integrating factor is $\mu(t) = e^{\int \frac{1}{250}dt} = e^{\frac{1}{250}t}$.

The general solution is, then,

$$Q = \frac{1}{e^{t/250}} \int 8e^{\frac{t}{250}}\,dt = e^{\frac{-t}{250}}\left(2000e^{\frac{t}{250}} + C\right) = 2000 + Ce^{\frac{-t}{250}}\ .$$

Applying the initial condition $Q(0) = 0$,

$$Q(0) = 0 = 2000 + C \qquad \rightarrow \qquad C = -2000.$$

Consequently, the particular solution for this problem is

$$Q(t) = 2000 - 2000e^{\frac{-t}{250}}.$$

The concentration of chlorine in the pool is given by the expression

$$\frac{Q(t)}{S(t)} = \frac{2000 - 2000e^{\frac{-t}{250}}}{1000} = 2 - 2e^{\frac{-t}{250}}.$$

Set the expression to equal 1, and solve for t. The time (in minute) it takes for the chlorine concentration in the pool to reach 1 gram per m^3 is:

$$1 = \frac{Q(t)}{S(t)} = 2 - 2e^{\frac{-t}{250}} \quad \rightarrow \quad -1 = -2e^{\frac{-t}{250}}$$

$$\rightarrow \quad \frac{1}{2} = e^{\frac{-t}{250}} \quad \rightarrow \quad \ln\left(\frac{1}{2}\right) = \frac{-t}{250}$$

$$\rightarrow \quad 250\ln\left(\frac{1}{2}\right) = -t \quad \rightarrow \quad t = 250\ln(2).$$

Non-constant-volume mixing problem (w/ constant rates, but $r_i \neq r_o$)

Example (*Exam 1, summer 2002*): A 400-liter tank is initially filled with 100 liters of dye solution with a dye concentration of 5 grams/liter. Pure water flows into the tank at a rate of 3 liters per minute. The well-stirred solution is drained at a rate of 2 liters per minute. Find the concentration of dye in the tank at the time that the tank is completely filled.

In this problem, $r_i = 3$, $r_o = 2$, and the initial volume is $S_0 = 100$. So the solution's volume is $S(t) = S_0 + (r_i - r_o)t = 100 + t$. The initial concentration of the solute is 5 grams per liter. Multiplying it by the initial volume gives us the initial condition of $Q(0) = 500$ (grams of dye). No number is given for the inflow concentration c_i, but it can be seen that $c_i = 0$ (why?). At the start, the 400-liter tank still has 300 liters of spare capacity left. At the rate of 1 liter net gain of content per minute, it can last 300 minutes until it is fully filled, so $t_{overflow} = 300$. Therefore, we can set up the initial value problem, for $t \leq 300$ (beyond that time, the mixing process will be of a different nature!):

$$Q' + \frac{2}{100+t}\,Q = 0 \quad , \qquad Q(0) = 500.$$

Where $p(t) = \dfrac{2}{100+t}$, and $\quad g(t) = 0$.

The integrating factor is

$$\mu(t) = e^{\int \frac{2}{100+t}dt} = e^{2\ln|100+t|} = e^{\ln(100+t)^2} = (100+t)^2.$$

The general solution is

$$Q = \frac{1}{(100+t)^2}\int 0\,dt = \frac{C}{(100+t)^2}.$$

Applying the initial condition $Q(0) = 500$,

$$Q(0) = 500 = \frac{C}{(100+0)^2} = \frac{C}{10000} \qquad \rightarrow \qquad C = 5000000.$$

Hence, the particular solution is $Q(t) = \dfrac{5000000}{(100+t)^2}$.

The problem asks for the concentration of dye at $t_{overflow} = 300$ minutes. Write down the formula for the solute's concentration and then set $t = 300$ to obtain

$$\frac{Q(300)}{S(300)} = \frac{5000000/(100+300)^2}{(100+300)} = \frac{5000000}{400^3} = \frac{5000000}{64000000} = \frac{5}{64}$$

$= 0.078125.$

Exercises A-1.4:

1. A tank is initially filled with 600 liters of a solution containing 100 grams of sugar. Solution containing a concentration of 2 g/liter sugar enters the tank at the rate 4 liters/minute and the well-stirred mixture leaves the tank at the same rate. Find the amount of sugar in the tank at time t, and find the limiting amount of sugar in the tank as $t \to \infty$.

2. A swimming pool holds 100 m^3 of pure water. Solution containing 2 kg/m^3 of chlorine enters the pool at a rate of 3 m^3/min. A drain is opened at the bottom of the pool so that the volume of solution in the pool remains constant. Find : (i) the amount of chlorine in the pool at time t, (ii) the amount of chlorine in the pool after one hour, and (iii) find the maximum amount of chlorine in the pool if the process is to continue indefinitely.

3. A 200-liter tank is filled to capacity with brine containing 1 g/liter of salt. Additional brine containing 5 g/liter of salt enters the tank at the rate 2 liters/min and the well-stirred mixture leaves the tank at the rate of 4 liters/min. Find the amount of salt in the tank at any time t, until the tank is completely drained ($0 < t < 100$). What is the maximum amount of salt present in the tank during this period?

4. A 150-liter mixing vat is initially filled with 60 liters of water containing 2 g/liter of dissolved potassium chloride. Starting at $t = 0$, 5 g/liter solution of potassium chloride flows into the vat at a rate of 6 liters/minute. The well-mixed solution leaves the vat at a rate of 3 liters/minute. (i) Set up an initial value problem describing the amount of potassium chloride in the vat at any time t prior to overflow, $0 < t < 30$. (ii) Solve this problem. (iii) Find the amount and concentration of potassium chloride in the vat at the time of overflow. (iv) Suppose the intake pipe continues to supply 6 liters/minute of solution past the time of overflow, and the excessive solution spills over the open top of the vat. Therefore, the well-mixed solution would leave at a rate of 6 liters/minute by means of both the output pipe and spill-over. Set up an initial value problem describing the amount of potassium chloride in the vat at any time t from the time of overflow onward, $t > 30$. (v) Solve this second initial value problem. (vi) Find the limiting concentration of potassium chloride in the vat as $t \to \infty$.

5. A retention pond initially contains 2000 m^3 of water having a pollutants concentration of 0.5 kg/ m^3. Each hour, 10 m^3 of water containing pollutants of variable concentration $2 + \sin(t)$ kg/ m^3 flows into the pond. Thoroughly mixed water flows out of the pond at the same rate. (i) Set up an initial value problem modeling this process. (ii) Solve this problem.

Answers A-1.4:

1. $Q(t) = 1200 - 1100e^{-t/150}$, 1200 grams
2. (i) $Q(t) = 200 - 200e^{-3t/100}$, (ii) $Q(60) = 200 - 200e^{-9/5} \approx 166.94$ kg, (iii) Q_{max} approaches 200 kg, occurs as $t \to \infty$.
3. $Q(t) = -0.08t^2 + 6t + 200$, $Q_{max} = 312.5$ grams (at $t = 37.5$)
4. (i) $Q' + \dfrac{1}{20+t}Q = 30, \qquad Q(0) = 120$

   (ii) $Q(t) = \dfrac{15t^2 + 600t + 2400}{20+t}$

   (iii) Amount of KCl = 678 grams, concentration = 4.52 g/liter

   (iv) $Q' + \dfrac{1}{25}Q = 30, \quad t > 30, \qquad Q(30) = 678$

   (v) $Q(t) = 750 - 72e^{(30-t)/25}$

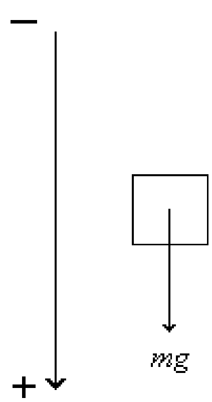
   (vi) 5 g/liter
5. (i) $Q' + \dfrac{1}{200}Q = 20 + 10\sin(t) \qquad , Q(0) = 1000$

   (ii) $Q(t) = 4000 + \dfrac{40000}{40001}\left(\dfrac{1}{200}\sin(t) - \cos(t)\right) - \dfrac{119963000}{40001}e^{\frac{-t}{200}}$

II. Air-resistance / Motion of an object in a resistive fluid medium

Freefall and Air-resistance

An object of mass m is freefalling near sea level (therefore, assume constant gravity). Unlike in the calculus class earlier, we will include the effect of air-resistance in our consideration. For the time being, we shall assume that the resistive force (drag force) is proportional to the instantaneous *speed** of the object in motion (e.g., when the resistance is due to friction only). Find the velocity of the freefalling object as a function of time.



Note: The textbook's convention is that the downwards direction is positive.

Forces acting on the object undergoing freefall:

Gravitational force	$w = mg$	(always downwards)
Resistive force (drag)	$\lvert F_r \rvert = k \lvert v \rvert$	(against the direction of v)

The gravity/weight is always downward, so w is always positive. The drag force always opposes the direction of the motion (given by the sign of velocity function $v(t)$). Therefore, $F_r = -kv$, which is always opposite of v but whose magnitude equals $k|v|$. The proportionality constant k is the *drag coefficient*.

* Speed = the magnitude of velocity = $|v|$

By Newton's second law of motion

$$\boldsymbol{ma = \sum forces}$$

That is $$ma = mv' = mg + (-\,kv).$$

Hence the required equation of motion governs the velocity of the object is

$$\boxed{mv' = mg - kv.}$$

It is a simple first order linear equation, with constant coefficients. Its solution is

$$v(t) = \frac{mg}{k} + Ce^{\frac{-k}{m}t}.$$

The position function of the motion can be found, as usual, by integration:

$$x(t) = \int v(t)\,dt.$$

Limiting velocity:

$$\boxed{v_L = \lim_{t\to\infty} v(t)}$$

The limiting velocity is the maximum velocity achievable by the object, in this model, given infinite amount of time to accelerate. Take the limit of the solution found above, we obtain $v_L = mg/k$.

More easily, it could also be found, without having to find $v(t)$ first, by setting $v' = 0$ in the original motion equation and solve for v. (Since v_L is the maximum velocity, it occurs at a critical point of $v(t)$! Hence, $v' = 0$. Physically, this happens when the gravitational force and drag cancel each other, leaving zero net force in the motion equation.)

That is

$$0 = mg - k v_L$$

$$mg = k v_L$$

Therefore, $$v_L = \lim_{t\to\infty} v(t) = \frac{mg}{k}.$$

Note that v_L is independent of any initial condition.

What happens if the initial velocity, for whatever reason, is larger than v_L? In that case the right hand side of the motion equation, $mg - kv$, is negative. The process modelled becomes a gradually decelerating motion whose velocity would eventually *slow down* to v_L, which would be the minimum achievable velocity.

Another (more realistic) air-resistance model

According to fluid dynamics, the drag force exerted on an object moving, sub-sonically, in a fluid (liquid or gas) medium is actually proportional to the square of its speed. Therefore, a more realistic equation that models the sub-sonic motion of an object in a resistive fluid medium is

$$\boxed{mv' = [\mathit{propulsive\ force}] - kv^2, \qquad v \ge 0}$$

This equation is a nonlinear first order differential equation. Fortunately, it is a separable equation. Therefore it is well within our capability to solve it.

Let $p > 0$ denotes the propulsive force (gravity, or the thrust of an engine, for examples) and $v \ne \pm\sqrt{p/k}$:

$$mv' = p - kv^2 \qquad \rightarrow \qquad \frac{m}{p - kv^2} v' = 1$$

Integrate both sides (the left can be integrated by partial fractions) to obtain the implicit solution:

$$\int \frac{m}{p - kv^2} dv = t + C$$

Even without an explicit function, the limiting velocity can nevertheless be found easily by setting $v' = 0$ in the motion equation.

$$v_L = \sqrt{\frac{p}{k}}$$

Example: A 100 *kg* Unmanned Aerial Vehicle (UAV) possesses propulsive force of 10000 *N* and has drag coefficient $k = 4$. Find the velocity function of its flight.

$m = 100$, $k = 4$, and take $v(0) = 0$ as the initial condition:

$$100v' = 10000 - 4v^2, \qquad v(0) = 0$$

Simplify the equation, separate the variables, and integrate both sides. (But first noting that $v = \pm 50$ are both also solutions of this equation.)

$$v' = 100 - 0.04v^2, \qquad \frac{dv}{100 - 0.04v^2} = dt$$

$$\int \frac{dv}{100 - 0.04v^2} = \int dt\,.$$

The left-hand side could be simplified by partial fractions into:

$$\frac{1}{100 - 0.04v^2} = \frac{1/20}{10 + 0.2v} + \frac{1/20}{10 - 0.2v}\,.$$

Hence,

$$\frac{1}{20}\int \left[\frac{1}{10 + 0.2v} + \frac{1}{10 - 0.2v}\right] dv = t + C$$

$$\frac{1}{20}\int 5\left[\frac{0.2\,dv}{10 + 0.2v} - \frac{-0.2\,dv}{10 - 0.2v}\right] = t + C$$

$$\frac{1}{4}\left[\ln|10 + 0.2v| - \ln|10 - 0.2v|\right] = t + C\,.$$

Now use the initial value $v(0) = 0$ to find $c = 0$. Therefore,

$$t = \frac{1}{4}\left[\ln|10 + 0.2v| - \ln|10 - 0.2v|\right] = \frac{1}{4}\ln\left|\frac{10 + 0.2v}{10 - 0.2v}\right|.$$

The limiting velocity (forward) is $v_L = 50$ *m*/*sec*, which is found by setting $v' = 0$ in the original equation and solve for v.

With some algebra (and a little patient) we can also find the explicit solution for this problem without much difficulty.

For reasons we shall see very shortly (in the section on *autonomous equations*), and given the condition that $v \geq 0$, there are 3 families of solutions, depending on the initial v-value. Since $v_0 = 0$ in this example, we will only find the relevant solution, which exists on the interval $50 > v \geq 0$.

From the implicit solution

$$t = \frac{1}{4}\ln\left|\frac{10+0.2v}{10-0.2v}\right|, \qquad 50 > v \geq 0.$$

$$4t = \ln\left|\frac{10+0.2v}{10-0.2v}\right|$$

$$e^{4t} = \left|\frac{10+0.2v}{10-0.2v}\right|$$

Since $50 > v \geq 0$, we can drop the absolute value:

$$(10-0.2v)e^{4t} = 10+0.2v$$

$$10e^{4t} - 10 = 0.2e^{4t}v + 0.2v$$

$$10(e^{4t} - 1) = 0.2(e^{4t} + 1)v$$

$$50(e^{4t} - 1) = (e^{4t} + 1)v$$

$$v = \frac{50(e^{4t} - 1)}{e^{4t} + 1}$$

Verify that $v(0) = 0$ and $\lim_{t\to\infty} v(t) = 50 = v_L$. The constant function $v = 50$, by the way, is also a solution to the equation (verify this). It does not come from the implicit general solution found earlier, but rather comes about by merely setting $v' = 0$ and solve for v.

III. Continuous compound interest with additional transactions

From calculus: Starting with a fixed principal amount A_0, the balance of an account garnering a fixed interest rate r (per year, usually) compounding at a frequency of m (per year) is given by the formula

$$A(t) = A_0\left(1+\frac{r}{m}\right)^{mt}.$$

If the interest is compounded continuously, i.e. as the frequency $m \to \infty$, then

$$A(t) = \lim_{m\to\infty} A_0\left(1+\frac{r}{m}\right)^{mt} = A_0 e^{rt}.$$

Indeed, $A_0 e^{rt}$ is the actual solution of the differential equation $A' = rA$, subject to the initial condition $A(0) = A_0$. (Exercise: verify this claim.)

Note: The simple equation above, $A' = rA$, means simply that the rate of change of the account balance is (continuously) proportional to its present size. The same equation (where the rate of change of some quantity is directly proportional to the current size of the said quantity) also governs exponential growth and radio-active decay (when r is negative) behaviors.

Now, instead just let the principal sit untouched and allowed to grow exponentially during the lifetime of the deposit (as in a bank CD), we will consider the effect of further deposit/withdraw transactions after the initial deposit. One caveat: since we do not have the necessary tool (the Laplace Transform, chapter 6) to deal with discrete (one-time) events, we have to assume that the transactions occur continuously, or at least occur regularly and frequently enough that they can be thought of as to be occurring continuously. While such an assumption does not model well the account balance of a typical checking account, it does give a good approximation of accounts with fixed installment payments such as annuities, mortgage or student loan repayment, etc.

Hence, assume an account starts with a principal of A_0, that gains interest at a rate of r per unit time compounded continuously. In addition, transactions occurring continuously and netting k amount per unit time ($k > 0$ means a net deposit into the account; $k < 0$ means a net withdraw from it) are applied to the account. Then, the account balance is described by the following initial value problem:

$$y' = r\,y + k, \qquad\qquad y(0) = A_0.$$

> *Comment*: The above equation says that at any moment in time the account balance y is increased by an amount proportional to its current size times the interest rate, and the rate of change is further modified (up or down, depending on the sign of k) by the net transactional amount.

We have solved this equation earlier in the semester. The general solution is

$$y = \frac{C}{r}e^{rt} - \frac{k}{r}.$$

Apply the initial condition we get:

$$y(0) = A_0 = \frac{C}{r}e^{0} - \frac{k}{r} = \frac{C-k}{r}$$

$$rA_0 = C - k \qquad \rightarrow \qquad C = rA_0 + k$$

Therefore,

$$y = \left(A_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r} = A_0 e^{rt} + \frac{k}{r}\left(e^{rt} - 1\right).$$

Example (*final exam, fall 2007*): A college student borrows \$5000 to buy a car. The lender charges interest at an annual rate of 10%. Assume the interest is compounded continuously and that the student makes payments continuously at a constant annual rate k. Determine the payment rate k that is required to pay off the loan in 5 years.

In this problem, the yearly interest rate $r = 10\% = 0.1$, the principal balance is $A_0 = 5000$, the yearly payment (think it as a withdrawal, since we are paying *down* the balance) k is the unknown. The loan term is 5 years, i.e. it needs to be paid off completely in 5 years. That means besides the initial condition, $y(0) = 5000$, we also have a second (terminal?) condition of $y(5) = 0$. We set up the required initial value problem (note that k has a minus sign in front, denoting repayment):

$$y' = r\,y - k = 0.1y - k \qquad y(0) = 5000.$$

It is a first order linear equation (it is also a separable equation),

$$y' - 0.1y = -\,k,$$

where $p(t) = -\,0.1$, and $g(t) = -\,k$.

The integrating factor is, therefore, $\mu(t) = e^{\int \frac{-1}{10}dt} = e^{-t/10}$.

The general solution is

$$y = \frac{1}{e^{-t/10}} \int -k\,e^{-t/10}\,dt = e^{t/10} \int -k\,e^{-t/10}\,dt = e^{t/10}\left(10\,k\,e^{-t/10} + C\right)$$

$$= 10\,k + C\,e^{t/10}$$

Apply the initial condition to find $C = 5000 - 10k$. The particular solution is

$$y(t) = (5000 - 10k)\,e^{t/10} + 10k.$$

Lastly, apply the pay-off condition $y(5) = 0$, we find that

$$k = 500\frac{e^{1/2}}{e^{1/2} - 1} = 500 + \frac{500}{e^{1/2} - 1}.$$

Example: The *present value* of a lottery jackpot − A lucky college student has won the lottery's ten million dollars jackpot. The winning is paid out equally over 20 years. Assume the payout is made continuously and the annual interest rate is constant 8% over the 20-year period. How much is the jackpot worth in today's dollar?

The problem is to find the *present value* (or *discounted value*) of this jackpot, which is not paid out all at once, but over a period of 20 years. That is, if the winning was, instead, made in a lump sum and was immediately deposited in a bank to garner interest for 20 years, how big a sum it must be to equal in value, at the end, to the 20-year of steady cash stream? In accounting-speak, we are trying to discount the future cash flow in order to find its present value, or <u>how much this future stream of cash payments is worth now</u>.

There are 2 ways to tackle this problem. The more obvious (to us non-accounts) is the indirect approach. First we compute the worth of this jackpot at the end of 20 years by solving the compound interest equation with the yearly interest rate $r = 8\%$, the yearly payment k is $\$10{,}000{,}000 / 20 = \$500{,}000$, and (the initial condition) the principal balance $A_0 = 0$. Set $t = 20$ in the result to obtain the terminal value after 20 years. Then we solve a second problem of continuous compound interest with an unknown initial principal balance A_1, no additional transactions, and a terminal condition $y(20)$ equal to the amount we have found previously. Solve this second problem to find A_1, which is how much the jackpot would be worth presently.

There is another, more direct, way to find the present value. It is how accountants will approach this problem − from the point-of-view of the lottery administrator. For the administrator, the problem is to set aside enough money to be deposited in a bank account such that a yearly payout/withdraw of $k = -\$500{,}000$ can be made for 20 years (and the account balance becomes exactly zero at the end of the 20-th year). In this approach, we will use $r = 8\%$, the yearly withdraw $k = -\$500{,}000$, the initial condition being the unknown principal balance A_0, plus the terminal condition $y(20) = 0$.

Hence, we will solve the initial value problem:

$$y' = r\,y + k = 0.08\,y - 500000, \qquad y(0) = A_0,$$

and such that $\quad y(20) = 0.$

In its standard form,

$$y' - 0.08y = -\,500000,$$

with $p(t) = -0.08$, and $g(t) = -500000$.

The integrating factor is, therefore, $\mu(t) = e^{\int -0.08\,dt} = e^{-0.08t}$.

The general solution is

$$y = \frac{1}{e^{-0.08t}} \int -500000\,e^{-0.08t}\,dt = e^{0.08t} \int -500000\,e^{-0.08t}\,dt$$
$$= e^{0.08t}\left(6250000\,e^{-0.08t} + C\right) = 6250000 + C\,e^{0.08t}$$

Apply the initial condition to find $C = A_0 - 6250000$. The particular solution is, consequently,

$$y(t) = 6250000 + (A_0 - 6250000)\,e^{0.08\,t}.$$

Lastly, apply the terminal condition $y(20) = 0$, we find that the present value of this nominally ten million dollars jackpot is actually less than half of its stated amount:

$$A_0 - 6250000 = -\,6250000\,/\,e^{1.6} \qquad \rightarrow \qquad A_0 = \$4{,}988{,}147$$

This example explains why that, when a lottery winner chooses (as most of them do, given the option) to take the winning in a single lump sum, rather than in periodic payments over many years, the payout amount becomes much smaller than the quoted jackpot, even before taxes are deducted…

Exercises A-1.5:

1. A home-buyer is applying for a 30-year mortgage at a fixed rate of 6% per year. Suppose the home-buyer can afford to repay no more than $1000 per month. What is the maximum amount of mortgage that the home-buyer can borrow?

2. A mortgage of $250000 has a fixed interest rate of 6% per year compounded continuously. (i) How much would the monthly payment be if the mortgage is to be paid off in 15 years? How much interest would have been paid? (ii) How much would the monthly payment be if the mortgage is to be paid off in 30 years? How much interest would have been paid?

3. A reservoir initially contains 15000 fish. The fish population grows continuously at a rate of 10% per year. Suppose each year local anglers harvest a fixed quota of 1000 fish from the reservoir. (i) Write an initial value problem that models the reservoir's fish population. (ii) Solve the initial value problem. (iii) How many fish will there be after 10 years?

4. The process of radioactive decay is described by the equation $y' = -r\,y$, where r is a positive constant, called the *decay constant* of the radioactive material. (i) Find an explicit formula for the material's half-life in term of r by first solving the equation together with the conditions $y(0) = \beta$ and $y(t_{half}) = \beta/2$. (ii) Given that the decay constant of uranium-235 is $r = 9.84\times10^{-10}$ per year, find its half-life.

Answers A-1.5:

1. $166940.22
2. (i) Repayment = $2106.4 per month ($25276.77 per year), total interest paid is $129151.5. (ii) Repayment = $1497.54 per month ($17970.50 per year), total interest paid is $289115.13.
3. (i) $P' - 0.1P = -1000, \qquad P(0) = 15000$
   (ii) $P(t) = 5000e^{t/10} + 10000$
   (iii) $P(10) = 5000e + 10000 \approx 23591$
4. (i) $t_{half} = \dfrac{\ln 2}{r}$, (ii) $t_{half} = 7.04\times10^{8}$ years.