THE (SIN/COS) FOURIER SERIES

A function $f(x)$ that is periodic, i.e.

$$f(x+p) = f(x) \quad \forall x$$

(p, a constant is called the period. Let $p = 2\ell$)

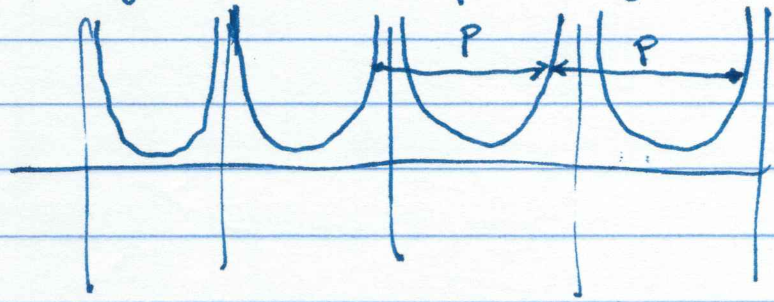
is "square integrable":

$$\int_x^{x+p} |f(s)|^2 ds < \infty$$

is at least piece-wise continuous (and has right & left hand derivatives at all x) has a Fourier Series (FS)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$$ //

An example of a non square integrable periodic function:



It's integral is undefined, for $\int_x^{x+p}$ //

Take $f(x) = f_e(x) + f_o(x)$

where $f_e(x)$ is the ~~even~~ even "part" and $f_o(x)$ is the odd "part" of $f(x)$

i.e. $\begin{cases} f_e(-x) = f_e(x) \\ f_o(-x) = -f_o(x) \end{cases}$

$$(*)\quad f(x) = f_e(x) + f_o(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$$

To find a_n's & b_n's, multiply both sides by $w(x)=1$ and $\left\{\begin{matrix}\sin \\ \cos\end{matrix}\right.\left(\frac{m\pi x}{\ell}\right)$ & integrate: first, $\cos\frac{m\pi x}{\ell}$ case

$$\int_{-\ell}^{\ell} f_e \cos\frac{m\pi x}{\ell}\,dx + \cancelto{0}{\int_{-\ell}^{\ell} f_o \cos\frac{m\pi x}{\ell}\,dx}$$

$$= a_0\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\,dx + \sum_{n=1}^{\infty} a_n \int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\cos\frac{n\pi x}{\ell}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\sin\frac{n\pi x}{\ell}\,dx$$

since $\begin{cases} \cos\frac{m\pi x}{\ell} \text{ are all even } & m=0,1,2\ldots \\ \sin\frac{m\pi x}{\ell} \text{ are all odd } & m=1,2,3\ldots \end{cases}$

$$\int_{-\ell}^{\ell} f_e \cos\frac{m\pi x}{\ell}\,dx = \sum_{n=1}^{\infty} a_n \int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\cos\frac{n\pi x}{\ell}\,dx$$

$$+ a_0 \int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\,dx$$

for $n=m=1,2,3\ldots$

$$\int_{-\ell}^{\ell} f_e(x)\cos\frac{m\pi x}{\ell}\,dx = a_m \int_{-\ell}^{\ell} \cos^2\frac{m\pi x}{\ell}\,dx = a_m \ell$$

$$(\star)\quad \therefore\ a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f_e(x)\cos\frac{m\pi x}{\ell}\,dx = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{m\pi x}{\ell}\,dx$$

$m=1,2,3\ldots$

for $n=m=0$

$$\int_{-\ell}^{\ell} f_e(x)\,dx = a_0 \int_{-\ell}^{\ell} dx = 2\ell a_0$$

$$(\star\star)\quad \therefore\ a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f_e(x)\,dx = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

Now $\sin\frac{m\pi x}{l}$ case:

$$\int_{-l}^{l} \cancel{f_e \sin\frac{m\pi x}{l}}^{\,0} dx + \int_{-l}^{l} f_o \sin\frac{m\pi x}{l}\,dx = a_0\int_{-l}^{l}\sin\frac{m\pi x}{l}\,dx$$

$$+\sum_{n=1}^{\infty} a_n \int_{-l}^{l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$+\sum_{n=1}^{\infty} b_n \int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$\int_{-l}^{l} f_o \sin\frac{m\pi x}{l}\,dx = \sum_{n=1}^{\infty} b_n \int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

for $m=n=1,2\ldots.$

$$\int_{-l}^{l} f_o(x)\sin\frac{m\pi x}{l}\,dx = b_m\int_{-l}^{l}\sin^2\frac{m\pi x}{l}\,dx = b_m\, l$$

(☆☆☆) $\therefore\ b_m = \frac{1}{l}\int_{-l}^{l} f_o(x)\sin\frac{m\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}\,dx$

note: $m=0$ contributes nothing.

So: $f(x)$ has a series representation $(\bigstar)$ where the a_0, a_n, b_n can be computed using $(\bigstar)$, $(\bigstar\bigstar)$ & $(\bigstar\bigstar\bigstar)$

Also, if $f(x) = f_{even}(x)$ then $f(x)$ will only have a cosine series

if $f(x) = f_{odd}(x)$ then $f(x)$ will only have a sine series.

- The fourier series of a periodic square-integrable function converges:

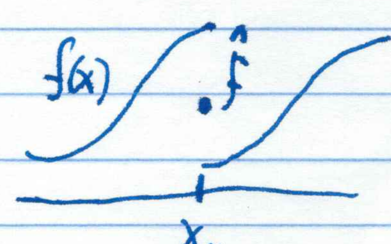
The Fourier series of $f(x)$ converges to $f(x)$, when $f(x)$ is continuous.

The series converges to

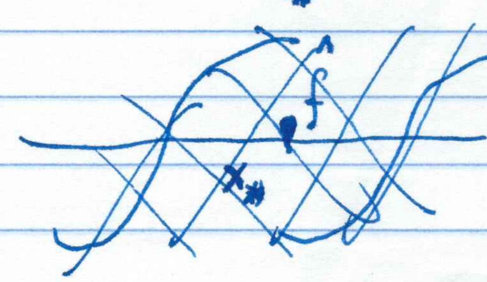
$$\frac{1}{2}\left(f(x_*^-) + f(x_*^+)\right) = \hat{f}$$

at some point $x = x_*$ where $f(x)$ is not continuous.

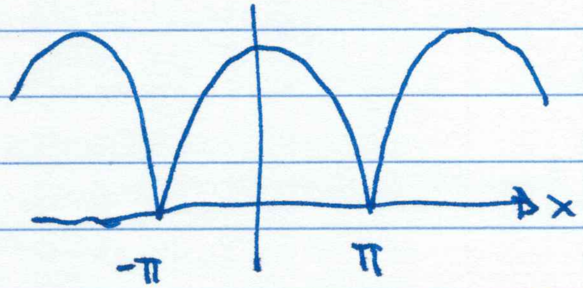
$$f(x_*^-) = \lim_{h \to 0} f(x_* - h)$$



$$f(x_*^+) = \lim_{h \to 0} f(x_* + h)$$



ex) find Fourier Series of $\cos^2 x$ in $-\pi \le x < \pi$



here $p = 2l = 2\pi$

$l = \pi$

$\cos^2(x+2\pi) = \cos^2 x$ periodic

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2x)\, dx = \pi \therefore \text{ square integrable.}$$

also $f(x) = \cos^2 x$ is continuous.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

but $f(x)$ is even $\therefore b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \,.$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2x) \cos nx \, dx$$

$$a_n = \overbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cos nx\,dx}^{0} + \frac{1}{\pi}\frac{1}{2}\int_{-\pi}^{\pi} \cos 2x \cos nx\,dx$$

unless $n=2$, this integral is zero by orthogonality

$$a_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 2x\,dx = \frac{1}{2}$$

$\therefore\ f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ (a finite Fourier series)

We derived the trig identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
That's because the trig identity is already in FS form!

Def: the finite Fourier Series (FFS)

$$S_N = a_0 + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

Def: $$\|g(x)\|_2 = \sqrt{\int_x^{x+?} |g(s)|^2 ds}$$

is the L_2 norm (for $g(x)$ periodic).

if $g(x)$ is square integrable, $\|g\|_2 < \infty$.

We said a Fourier Series converges. But in what sense?

If $f(x)$ in L_2 periodic

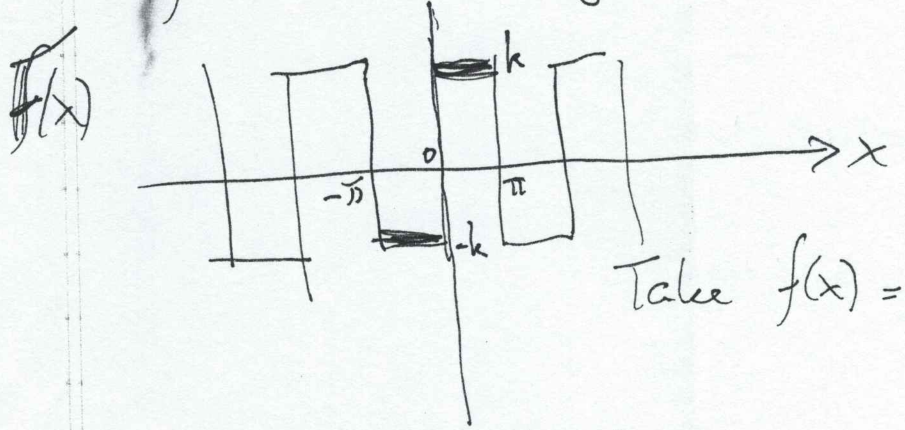
$$\| f(x) - S_N(x) \|_2^2 < \varepsilon_N$$

where ε_N is a non-negative number that depends on N, the number of terms in the FFS. In fact

$$\lim_{N\to\infty} \| f(x) - S_N(x) \|_2^2 = 0$$

Which is to say that the FS converges to $f(x)$ in the L_2-norm.

ex) This is an example worked out in the book



A square wave.

Take $f(x) = \begin{cases} -k & -\pi \le x < 0 \\ k & 0 \le x < \pi \end{cases}$

$f(x+2\pi) = f(x)$, clearly.

& also square integrable:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = k^2 \int_{-\pi}^{0} dx + k^2 \int_{0}^{\pi} dx = 2\pi k^2 < \infty .$$

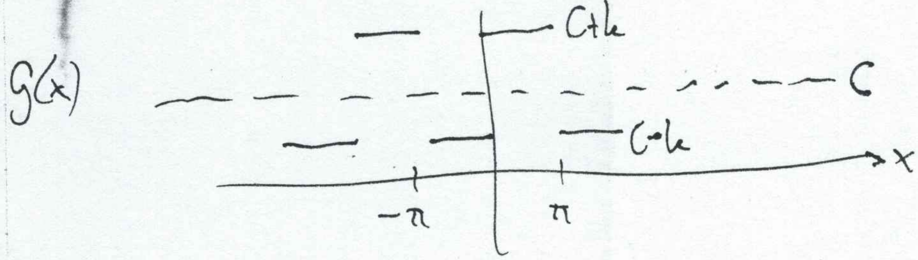
it is piece-wise continuous with discontinuities at $x = -\pi, 0, \pi \ldots$

$f(x) = -f(-x)$; ie. ODD. So expect

$a_n = 0 \quad \forall n \geq 1$

What about a_0? , $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$, the mean value of $f(x)$, so clearly a_0 must be 0.

Aside: Suppose knew Fourier Series for $f(x)$, as above, and wanted FS of $g(x)$



So series of $g(x)$ would be the SAME as $f(x)$, but just with the addition of C, ie.

$$g(x) = C + FS(f(x))$$

So $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n=1,2\ldots$$

$$b_n = \frac{1}{\pi}\left[-\int_{-\pi}^{0} k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] = \frac{2k}{n\pi}(1-\cos nx)$$

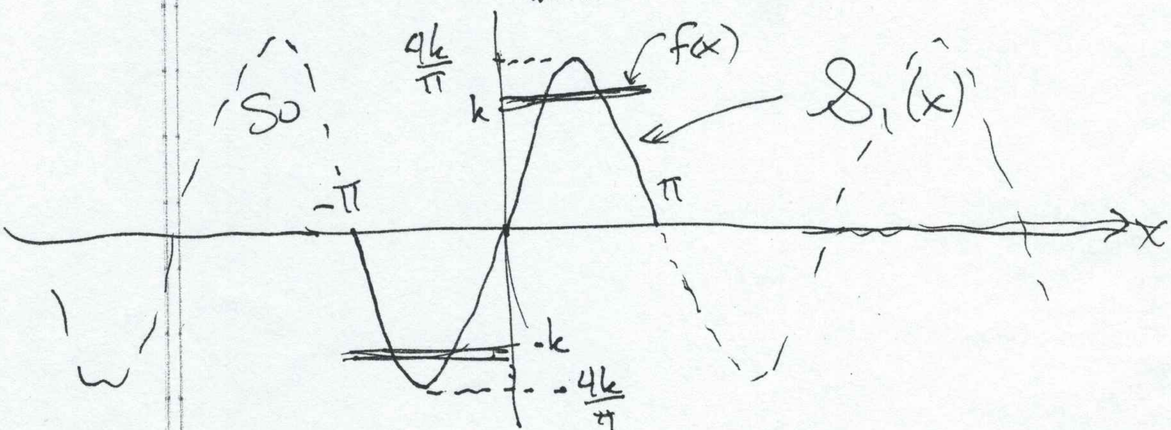
but $$(1-\cos nx) = \begin{cases} 2 \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even} \end{cases}$$

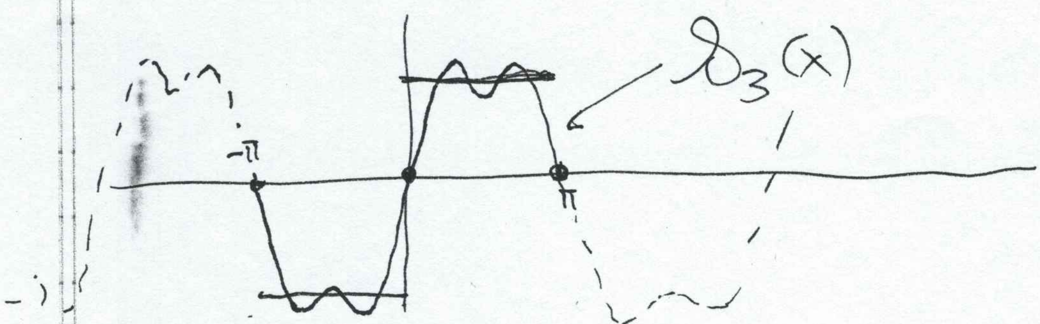
so $b_1 = \frac{4k}{\pi}$ $b_2 = 0$ $b_3 = \frac{4k}{3\pi}$ $b_4 = 0$, $b_5 = \frac{4k}{5k}$ ....

$$\text{so } f(x) = \sum_{m=0}^{\infty} \frac{4k}{\pi} \frac{1}{(2m+1)} \sin[(2m+1)x] \quad (\pounds)$$

$$= \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots$$

$$\text{let } S_N = \sum_{m=0}^{N} \frac{4k}{\pi} \frac{1}{(2m+1)} \sin[(2m+1)x]$$





etc

so the larger N, the closer it gets to $f(x)$.

Also, at $0, \pi, -\pi$, etc $\sum_{m=0}^{\infty} \frac{4k}{\pi} \frac{1}{(2m+1)} \sin[(2m+1)x] = 0$!

i.e. the average $\frac{1}{2}[f(0^-) + f(0^+)]$, $\frac{1}{2}[f(\pi^-) + f(\pi^+)]$, etc.

Rewrite $(\mathcal{L})$ as

$$f(x) = A \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin[(2m+1)x]$$

$$A = \frac{4k}{\pi}.$$

- we note that the wave number $n = \frac{2\pi}{\lambda_n}$ $n = 1, 3, 5, 7 \ldots$

or $2m+1 = \frac{2\pi}{\lambda_m}$ $m = 0, 1, 2, \ldots$

increases then the wavelength λ_n (or λ_m) decreases.

- We note that $\max_{-\pi < x < \pi} \left| \sin[(2m+1)x] \right| = 1$

i.e. all sines have maximum amplitude 1.

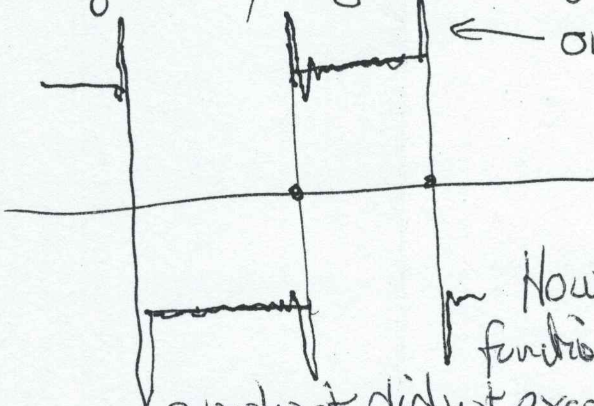
- Hence the size of the b_n's here drops $\propto \frac{1}{n}$, roughly.

~~So taking~~ So the $n =$ 101 (or $m = 50$) contribution to the sum is about 100 times smaller than the first term in the series, roughly.

- The sum in $(\mathcal{L})$ takes a bunch of sine waves, adds them up with the right amplitude & you get something that looks like $f(x)$. The more terms, the better.

BUT there's a rub...

If you take S_N for $f(x)$ above, for N getting really large, we get



it gets more localized but worse as $N \to \infty$

However, for piecewise constant functions Gibbs showed that the overshoot did not exceed more that about 18%

This is called the "Gibbs Phenomenon". You'll see this happen at discontinuities of $f(x)$.

OK, so the series (†) doesn't converge to $f(x)$, as $N \to \infty$?? See (⁂) above.

It does, everything is OK. It converges to $f(x)$ if you SPECIFICALLY measure the discrepancy between $f(x)$ & $S_N(x)$ as $N \to \infty$ globally, an in the L_2-norm:

$$\lim_{N \to \infty} \| f(x) - S_N(x) \|_2^2 = 0$$

in the L_2 norm the wiggles you get in the overshoot will "cancel out" in the integration

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|f(x)-\sum_{m=0}^{N}b_{2m+1}\sin\left[(2m+1)x\right]\right|^2 dx$$ giving 0.

We say that the rate of convergence for our square wave fourier series

$$\propto \frac{1}{n}$$

But if $f(x)$ was smooth, the FS will converge faster, at a rate $\propto \frac{1}{n^2}$

So adding 40 terms sort of improves the answer by 40^2... sort of. So pretty fast. The fair way to put this is the following:

if you have a series with 10 terms, adding an extra term contributes roughly $\frac{H}{10^2}$ in magnitude to the answer, where H is the typical size of each coefficient.

We can show this as follows:

Take $f(x)$ continuous, as are the derivatives of $f(x)$

Take $f(x)$ periodic, odd & L_2, with period 2π, for example.

$\cancel{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\pi x\,dx = -\frac{1}{n\pi}\overbrace{\cancel{f(x)\cos nx}}^{0}\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$$

by integrating by parts. The first term is always 0 because $\cos nx$ evaluate at $\pm\pi$.

Integrate by parts again:

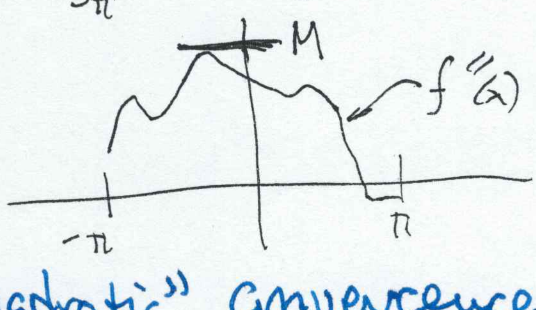
$$b_n = \overbrace{\cancel{\frac{f'(x)\sin nx}{n^2\pi}}}^{0}\Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\sin nx\,dx$$

we note that $\max_{-\pi<x<\pi}|\sin nx| = 1$ so

$$|+b_n| = \left|\frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\sin nx\,dx\right| \le \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)|\sin nx|\,dx$$

$$\le \frac{1}{n^2\pi}\int_{-\pi}^{\pi}|f''(x)|\,dx\int_{-\pi}^{\pi}|\sin nx|\,dx \le \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\,dx\right| \le \frac{2M}{n^2}$$

where

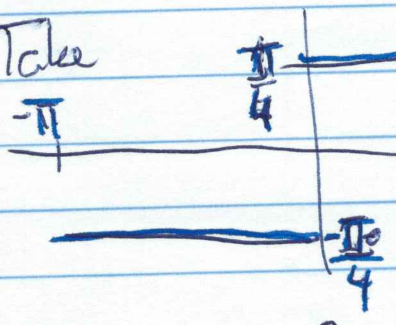


$f''(x)$ plot, so $|b_n| \propto \frac{1}{n^2}$ for $f(x)$ continuous

which is "quadratic" convergence, ...pretty fast.

Gibbs Phenomenon: recall the overshoot in the Fourier series of piecewise continuous functions? Why does it occur? Is the overshoot become worse as more terms are taken in the series?

Take



The series for this square wave is:

$$S_N(x) = \sin x + \frac{1}{3}\sin 3x + \dots + \frac{1}{N-1}\sin\left[(N-1)x\right]$$

subst $x=0$

$$S_N(0) = 0 = \frac{-\frac{\pi}{4}+\frac{\pi}{4}}{2} = \frac{f(0^-)+f(0^+)}{2}$$

Now, compute

$$S_N\left(\frac{2\pi}{2N}\right) = \frac{\pi}{2}\left[\frac{2}{N}\operatorname{sinc}\left(\frac{1}{N}\right) + \frac{2}{N}\operatorname{sinc}\left(\frac{3}{N}\right) + \dots \frac{2}{N}\operatorname{sinc}\left(\frac{[N-1]x}{N}\right)\right]$$

i.e. $\boxed{\operatorname{sinc} x = \frac{\sin x}{x}}$

since

$$S_N\left(\frac{2\pi}{2N}\right) = \sin\frac{\pi}{N} + \frac{1}{3}\sin\left(\frac{3\pi}{N}\right) + \dots \frac{1}{N-1}\sin\left[\frac{(N-1)\pi}{N}\right]$$

this is a midpoint numerical integration of $\int_0^1 \operatorname{sinc} x\,dx$ with spacing $2/N$:

$$\lim_{N\to\infty} S_N\left(\frac{2\pi}{2N}\right) = \frac{\pi}{2}\int_0^1 \operatorname{sinc} x\,dx = \frac{1}{2}\int_0^{\pi}\frac{\sin t}{t}\,dt = \frac{\pi}{4} + \frac{\pi}{2}(0.089490\dots)$$

so about 18% ~~over~~ overshoot

Similarly $\lim_{N\to\infty} S_N\left(\frac{-2\pi}{2N}\right) = -\frac{\pi}{4} - \frac{\pi}{2}(0.089490\dots)$

(Fourier) Bessel Expansions

Consider

$$x^2 y'' + x y' + (\mu^2 x^2 - \nu^2) y = 0 \qquad x > 0 \qquad (*)$$

$$\nu \geq 0$$

is an SL equation: if $p = x$, $\cancel{q = -\nu^2}$

$$q = -\frac{\nu^2}{x}$$

$$w = x$$

$$\lambda = \mu^2$$

Then $(*)$ is

$$\mathcal{L} y + \lambda w y = 0$$

//

The solution of $(*)$

$$y = \begin{cases} c_1 J_\nu(\mu x) + c_2 J_{-\nu}(\mu x) & \nu \notin \mathbb{Z} \\ c_1 J_\nu(\mu x) + c_2 Y_\nu(\mu x) & \nu \in \mathbb{Z} \end{cases}$$

The J_ν, $J_{-\nu}$ are called Bessel functions. They oscillate.

J_ν are bounded at $x = 0$

$J_{-\nu}$ are unbounded at $x = 0$.

Y_ν are Bessel functions of the "second kind", they oscillate and are unbounded at $x = 0$.

So, if B.C. $y(0) = a$, a number, possibly 0

then $y = c_1 J_\nu(\mu x)$

i.e. $c_2 \equiv 0$

(Supposing $0 \leq x \leq l$ is domain)

(a) if B.C. $y(l) = 0$

$J_\nu(\alpha_n) = 0$ $\quad \mu_n = \dfrac{\alpha_n}{l}$ $\quad n = 1, 2, 3 \ldots$

α_n are roots of J_ν

(b) if B.C. $y'(l) = 0$

$J'_\nu(\beta_n) = 0 \Rightarrow \mu_n = \dfrac{\beta_n}{l}$ $\quad n = 1, 2 \ldots$

β_n are roots of J'_ν

(c) if B.C. $k\,y(l) + l\,y'(l) = 0$ $\quad k \geq 0$

$k J_\nu(\gamma_n) + \gamma_n J'_\nu(\gamma_n) = 0$

$\mu_n = \dfrac{\gamma_n}{l}$ for $n = 1, 2 \ldots$

The J_ν, $J_{-\nu}$, Y_ν form orthogonal families of functions with respect to the weight $w(x) = x$ for $0 \leq x \leq l$

ex) Suppose $f(x)$ is defined $0 \leq x \leq l$

$f(x)$ is finite at $x = 0$

then $$f(x) = \sum_{n=1}^{\infty} c_n J_\nu(\mu_n x)$$

so
$$c_n = \frac{1}{Q} \int_0^l x f(x) J_\nu(\mu_n x)\, dx$$

where $Q = \int_0^l x J_\nu^2(x)\, dx$ //