DIMENSIONAL ANALYSIS
(AND SCALING)

Very useful and simple strategy that yields insights into physical processes as well as other processes with structure.

In addition to crucial insights into the physics of a problem it suggest ways to understand these in a simpler way.

Famously, G.I. Taylor in the 40's estimated the yield of the first atomic explosion from photos of the spread of the fireball (simplified version) of other atomic tests. Let's make the estimate ourselves, with the following simplifying assumptions:

- assume radial symmetry
- the background temperature and pressure are too small to impact the explosion.
- assume a point source.

The blast generates a powerful shock (pressure) wave. Let

$$g(t, r, \rho, e) = 0$$

air

t = time $[T]$

r = length $[L]$

ρ density of air $[M L^{-3}]$

e energy $\left[\frac{ML}{T^2} L\right] = [M L^2 T^{-2}]$

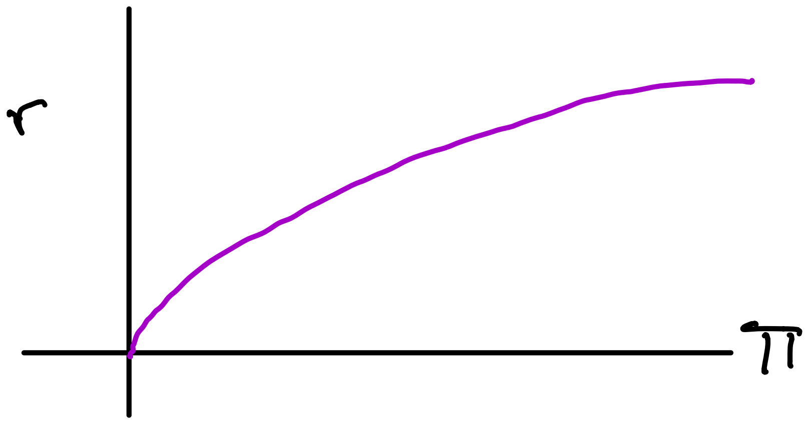
Note that $\dfrac{r^5 \rho}{t^2 e} = \dfrac{[L^5][M L^{-3}]}{[T^2][M L^2 T^{-2}]} = [1]$

we'll call this a "dimensionless group." Then

$$\therefore \quad \frac{r^5 \rho}{t^2 e} = C, \text{ a constant}$$

We solve for r and get that

$$r = C\left(\frac{e t^2}{\rho}\right)^{1/5} = C\,\Pi^{1/5} = g(\Pi)$$



Rmk: the constant C depends on the ratio of the specific heat at constant pressure to the specific heat at constant volume.

In other words there is a "law" that underlies the blast

$$f(\Pi) = f\left(\frac{r^5\rho}{t^2 e}\right) = 0$$

this is a function that could perhaps be found experimentally.

The basic rule of dimensional analysis is

DIMENSIONAL HOMOGENEITY

The underlying principle is that any law that models a physical process accurately must be independent of the units used to measure the physical variables.

So, among other things, the units on both sides of an equation should be the same:

$$F = ma$$

$$[Kg\, m/s^2] = [Kg\, m/s^2]$$

Rule: A function $f(x,y)$ is said to be homogeneous in x & y if

$$f(\beta x, \beta y) = \beta^{\alpha} f(x,y)$$

The Buckingham Π Theorem

A general form of a physical law

$$(\star) \quad f(q_1, q_2, \ldots, q_m) = 0$$

where q_i are dimensional quantities.

By inspection or using the theorem we will try to express $(\star)$ in terms of dimensionless groupings $\Pi_1, \Pi_2, \ldots, \Pi_{m-r}$ such that $(\star)$ can be expressed in unit-dimension form

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{m-r}) = 0$$

"dimensionless physical law"

How do we find Π_i's & F ? Buckingham Pi Theorem. If problem is simple enough, we can use inspection.

There are physical dimensions $L_1, L_2, \ldots, L_n$
$(n \leq m)$

e.g. $g(t, r, \rho, e) = 0$ in previous problem

$m = 4$ (# of q's)

$n = 3$ (# of L's) M, L, T

$\Pi_{m-r} = \Pi_{4-r} = \Pi_1 = \dfrac{r^5 \rho}{t^2 e}$

it so happens that $r = n$ in this case

So generally, we write

$$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \ldots L_n^{a_{ni}}$$

here $i = 1, 2, \ldots, m$

For all of the q's there is a matrix A

of the form

$$\begin{array}{cc} L\text{'s} & \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \\ & q \end{array} = A$$

"the dimension matrix"

This matrix is generally rectangular

THIS MATRIX A has rank(A) = r

Rmk: When there several alternative groupings the best choice encodes the scalings that best best represent the range of units of variables, relevant to the application of the model.

Also non-dimensionalizing leads to simpler physical laws.

ex) An example of homogeneisation:

$$f(x,t) = x - \frac{1}{2} g t^2 = 0$$

$$x = [L] \qquad t = [T] \qquad g = [LT^{-2}]$$

Define dimensionless quantities:

let $\bar{x} = \lambda_1^{-1} x \quad \bar{t} = \lambda_2^{-1} t$ where $\begin{cases} \lambda_1 = [L] \\ \lambda_2 = [T] \end{cases}$

$$\bar{g} = \lambda_1^{-1} \lambda_2^{+2} g$$

$$f(x,t) = x - \tfrac{1}{2} g t^2 = \lambda_1 \bar{x} - \tfrac{1}{2} (\lambda_1 \lambda_2^{-2}) \bar{g} \lambda_2^2 \bar{t}^2$$

$$= \lambda_1 (\bar{x} - \tfrac{1}{2} \bar{g} \bar{t}^2)$$

$$\therefore \quad f(x,t) = \lambda_1 f(\bar{x}, \bar{t})$$

$\therefore$ $f(x,t)$ is a homogeneous relation

$f(\bar{x}, \bar{t})$ is dimensionless

//

ex) A diffusion problem: at $t = 0$ an amount of heat energy e concentrated at a point in space is allowed to diffuse outward into a region with temperature 0. If r denotes the radial distance from the source at time t, the problem is to find the temperature $u(r,t)$.

(*) t, r, e affect $u(r,t)$

the rate at which heat diffuses is important

the medium in which the heat diffuses is also important.

c is the heat capacity $[E/K/L^3]$

k is the thermal diffusivity $[L^2/T]$

k is the thermal conductivity per unit of heat capacity, or amount of heat energy flowing across a unit area/unit time at a given temperature per unit heat capacity.

$$f(t, r, u, e, k, c) = 0$$

$t = [T]$ $\quad e = [E]$

$r = [L]$ $\quad k = [L^2 T^{-1}]$

$u = [K]$ $\quad c = [E/K/L^3]$

temperature

here $\left\{ \begin{array}{l} m = 6 \\ n = 4 \end{array} \right.$

$$\begin{array}{c|cccccc} & t & r & u & e & k & c \\ T & 1 & 0 & 0 & 0 & -1 & 0 \\ L & 0 & 1 & 0 & 0 & 2 & -3 \\ K & 0 & 0 & 1 & 0 & 0 & -1 \\ E & 0 & 0 & 0 & 1 & 0 & 1 \end{array} = A \text{ Dimension Matrix}$$

$\text{Rank}(A) = r = 4$ $\qquad m - r = 2 \therefore$

2 dimensionless groups.

Rmk: see classnotes in MTH341 to learn how to compute the rank of a matrix.

We now find the 2 dimensionless groups

Π_1, Π_2

$$1 = [\Pi] = [t^{\alpha_1} r^{\alpha_2} u^{\alpha_3} e^{\alpha_4} k^{\alpha_5} c^{\alpha_6}]$$

$$= T^{\alpha_1} L^{\alpha_2} K^{\alpha_3} E^{\alpha_4} (L^2 T^{-1})^{\alpha_5} (E K^{-1} L^{-3})^{\alpha_6}$$

Get
$$\begin{cases} \alpha_1 - \alpha_5 = 0 \\ \alpha_2 + 2\alpha_5 - 3\alpha_6 = 0 \\ \alpha_3 - \alpha_6 = 0 \\ \alpha_4 + \alpha_6 = 0 \end{cases}$$

Compare this to the A matrix entries

$$\begin{cases} \alpha_1 = -\frac{1}{2} \quad \alpha_2 = 1 \quad \alpha_3 = \alpha_4 = 0 \quad \alpha_5 = -\frac{1}{2} \quad \alpha_6 = 0 \\ \alpha_1 = \frac{3}{2} \quad \alpha_2 = 0 \quad \alpha_3 = 1 \quad \alpha_4 = -1 \quad \alpha_5 = \frac{3}{2} \quad \alpha_6 = 1 \end{cases}$$

$$\Pi_1 = r t^{-1/2} k^{-1/2} = \frac{r}{\sqrt{kt}}$$

$$\Pi_2 = t^{3/2} u e^{-1} k^{3/2} c = \frac{uc}{e} (kt)^{3/2}$$

So $f(t, r, u, e, k, c) = 0$

is equivalent to

$$F(\Pi_1, \Pi_2) = 0$$

Solve for

$$\Pi_2 = g(\Pi_1)$$

we get

$$u = \frac{e}{c}(kt)^{-3/2} g\left(\frac{r}{\sqrt{kt}}\right) \quad //$$

ex) Consider particle constant mass m radially projected upwards from Earth's surface with initial speed V. Let R be the radius of Earth. Let $\hat{x}$ and $\hat{t}$ be the distance from Earth's surface to particle, and time ≥ 0. Neglect drag

ODE
$$m \frac{d^2\hat{x}}{d\hat{t}^2} = -\frac{\hat{g}R^2}{(\hat{x}+R)^2} \quad (\text{\#})$$

I.C. $\begin{cases} \tilde{x}(0) = 0 \\ \dfrac{d\tilde{x}}{d\tilde{t}}(0) = V \end{cases}$

$\tilde{g} = \dfrac{GM}{R^2}$ the gravitational constant.

N.B. $m\dfrac{d^2\tilde{x}}{d\tilde{t}^2} = -G\dfrac{Mm}{(\tilde{x}+R)^2}$ is Newton's 2nd Law and reduces to ($\ddagger$)

Scale the problem:

$\tilde{x}$	$[L]$
$\tilde{t}$	$[T]$
$\tilde{g}$	$[LT^{-2}]$
V	$[LT^{-1}]$
R	$[L]$

$R \sim 6436$ Km

$$x \equiv \frac{\tilde{x}}{R}$$

$$t = \frac{\tilde{t}}{RV^{-1}}$$

$$\frac{d}{d\tilde{t}} = \frac{d}{dt}\frac{dt}{d\tilde{t}} = \frac{1}{RV^{-1}}\frac{d}{dt} = \frac{V}{R}\frac{d}{dt}$$

$$\frac{d^2}{d\tilde{t}^2} = \frac{d}{d\tilde{t}}\left(\frac{d}{d\tilde{t}}\right) = \frac{V^2}{R^2}\frac{d^2}{dt^2}$$

Replace these into ODE

$$\frac{V^2}{R^2}\frac{d^2(Rx)}{dt^2} = -\frac{\tilde{g}R^2}{(\tilde{x}+R)^2} = \frac{-\tilde{g}R^2}{(Rx+R)^2}$$

$$\frac{V^2}{R^2}\frac{d^2(Rx)}{dt^2} = \frac{-\tilde{g}R^2}{R^2(1+x)^2} = -\frac{\tilde{g}}{(1+x)^2}$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{-\tilde{g}R}{RV^2}\frac{1}{(1+x)^2}$$

let $\varepsilon \equiv \frac{V^2}{\tilde{g}R}$ then

$$(\$)\quad \begin{cases} \varepsilon \dfrac{d^2x}{dt^2} = -\dfrac{1}{(1+x)^2} \\ x(0) = 0 \\ \dfrac{dx(0)}{dt} = 1 \end{cases}$$

the solution of $(\$)$ is $x(\varepsilon, t)$

<u>Rmk:</u> R is an intrinsic length

$\frac{R}{V}$ is an intrinsic time

ε is a dimensionless parameter

if $\varepsilon \gg 1$

$$\frac{d^2x}{dt^2} = \frac{-1}{\varepsilon} \frac{1}{1+x^2}$$

then $\quad \dfrac{d^2x}{dt^2} \approx 0$

$\therefore \; x \sim v_0 t$ linear in time

$v \sim v_0$ constant

What are the conditions required for $\varepsilon \gg 1$?

We could have chosen a different scaling, one that use acceleration:

$$x = \frac{\hat{x}}{R} \qquad t = \frac{\hat{t}}{\sqrt{R/\hat{g}}}$$

$$\frac{d}{d\hat{t}} = \frac{d}{dt}\frac{dt}{d\hat{t}} = \frac{1}{\sqrt{R/\hat{g}}}\frac{d}{dt}$$ replace into original equation

$$\frac{R}{R/\hat{g}} \frac{d^2x}{dt^2} = \frac{-\hat{g}R^2}{R^2(1+x)^2}$$

or $$\frac{d^2x}{dt^2} = -\frac{1}{(1+x)^2}$$

$$x(0) = 0$$

$$\frac{R}{\sqrt{R/g}} \frac{dx}{dt}(0) = V \Rightarrow \frac{dx(0)}{dt} = \frac{V}{\sqrt{gR}} \equiv \varepsilon^{1/2}$$

$$\frac{d^2x}{dt^2} = -\frac{1}{(1+x)^2}$$

$$x(0) = 0$$

$$\frac{dx}{dt}(0) = \varepsilon^{1/2}$$

The story emphasizes the role of the initial conditions: if the escape velocity $\varepsilon^{1/2}$ is

$\varepsilon^{1/2} < 1$ vehicle does not escape Earth

$\varepsilon^{1/2} > 1$ vehicle escapes Earth

$\varepsilon = 0$ is the threhold.

We could have been more systematic:

$$1 = [\Pi] = [\hat{t}^{\alpha_1} \hat{x}^{\alpha_2} R^{\alpha_3} V^{\alpha_4} \hat{g}^{\alpha_5}]$$

$$1 = \bar{T}^{\alpha_1 - \alpha_4 - 2\alpha_5} L^{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$$

$$\therefore \quad \begin{cases} \alpha_1 - \alpha_4 - 2\alpha_5 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0 \end{cases}$$

Rank $r = 2$ $\quad m = 5$ expect 3 groupings:

$\therefore$

$$\Pi_1 = \frac{\bar{x}}{R} \qquad \Pi_2 = \frac{\hat{t}}{R/V} \qquad \Pi_3 = \frac{V}{\sqrt{\hat{g}R}}$$

hence $F(\Pi_1, \Pi_2, \Pi_3) = 0 \quad \therefore$

$\Pi_1 = h(\Pi_2, \Pi_3)$ some function of Π_2, Π_3

$$\frac{\bar{x}}{R} = h\left(\frac{\hat{t}}{R/V}, \frac{V}{\sqrt{\hat{g}R}}\right) \qquad (\#)$$

Suppose we want to find time t_{max} required to reach maximum height, given given some initial speed V:

$$\frac{\partial h}{\partial \Pi_2}\left(\frac{t_{max}}{R/V}, \frac{V}{\sqrt{gR}}\right) \equiv 0$$

$$\text{or} \quad \frac{1}{R}\frac{d\hat{x}}{dt}\bigg|_{t_{max}} = 0$$

$$t_{max} = \frac{R}{V} F\left(\frac{V}{\sqrt{gR}}\right)$$

the maximum heigh only depends on $\varepsilon^{1/2} = V/\sqrt{gR}$ a single parameter. F is a graph of

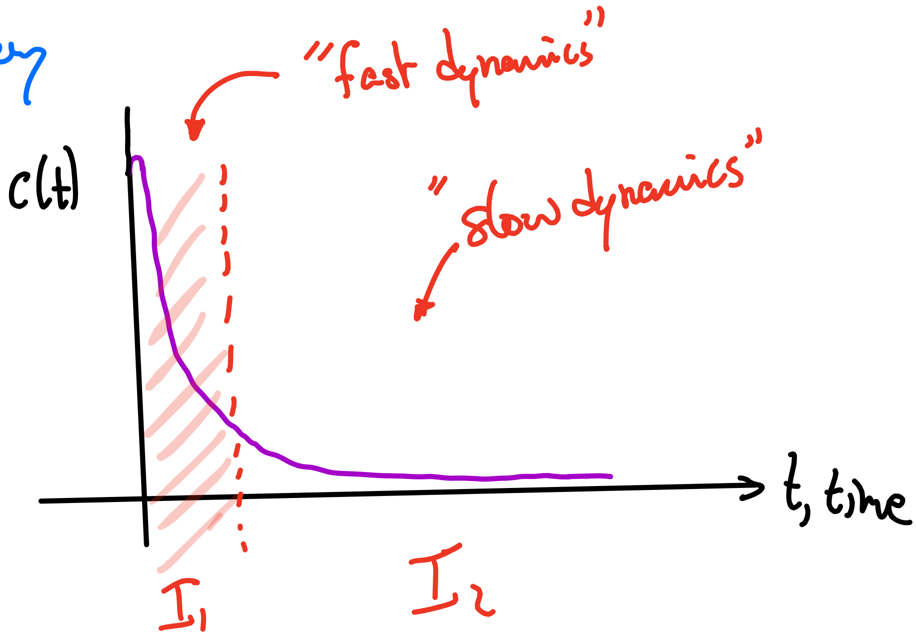
$$t_{max}/_{R/V} \quad \text{vs} \quad \varepsilon^{1/2}$$

contains all of the data required to find t_{max}, given the value of parameter $\varepsilon^{1/2}$ //

SCALING

Many processes that have inherently different typical scales.

In chemical reactions it is typical to see the evolution of a concentration $c(t)$ obey



I_1 is the fast reaction time
I_2 is the slower reaction time

Here the I_1 is the highly (time) dynamic regime and I_2 is the nearly (time) stationary regime

In I_1 we also have the reaction rate change quickly
in I_2, we don't.

ex) Consider projectile problem with the added dynamics of air drag:

$$\begin{cases} m\dfrac{d^2\hat{x}}{d\hat{t}^2} + k\dfrac{d\hat{x}}{d\hat{t}} = -\dfrac{GMm}{(\hat{x}+R)^2} \\ \hat{x}(0)=0 \\ \dfrac{d\hat{x}}{d\hat{t}}(0)=V \end{cases}$$

simple linear model: air drag is proportional to speed.

When $\hat{x}=0$ the gravitational force must be equal to $-m\hat{g}$

$$\therefore \frac{GMm}{R^2} = m\hat{g} \Rightarrow \hat{g} \text{ is so defined.}$$

Take $\hat{x} \ll R$

Expand the term

$$-\frac{1}{(\tilde{x}+R)^2} = -\frac{1}{R^2}\frac{1}{(1+\tilde{x}/R)^2}$$

Recall that

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \frac{p(p-1)(p-2)}{3!}z^3 + \cdots$$

for small z:

$$(1+z)^p \approx 1 + pz$$

we say that this is an $O(z^2)$ approximation

$$\therefore -\frac{1}{R^2(1+\tilde{x}/R)^2} = -\frac{1}{R^2}\left(1 - \frac{2\tilde{x}}{R}\right) + O\left(\left(\frac{\tilde{x}}{R}\right)^2\right)$$

(again, assuming $\tilde{x}/R << 1$)

For $\frac{\tilde{x}}{R} << 1$

$$m\frac{d^2\tilde{x}}{d\tilde{t}^2} + k\frac{d\tilde{x}}{d\tilde{t}} + \frac{GMm}{R^2(1+\tilde{x}/R)^2} = 0$$

$$= m\frac{d^2\tilde{x}}{d\tilde{t}^2} + k\frac{d\tilde{x}}{d\tilde{t}} + \frac{GMm}{R^2}\left(1 - \frac{2\tilde{x}}{R}\right) + O\left(\left(\frac{\tilde{x}}{R}\right)^2\right) = 0$$

To $O\left(\frac{\tilde{x}}{R}\right)$, with $\tilde{g} \equiv \frac{GM}{R^2}$

$$\approx m\frac{d^2\tilde{x}}{d\tilde{t}^2} + k\frac{d\tilde{x}}{d\tilde{t}} + m\tilde{g} = 0$$

$\tilde{x} = [L]$ $\quad\quad\quad$ $\tilde{g} = [LT^{-2}]$

$\tilde{t} = [T]$

$m = [M]$

what are the units of k?

Recall that $k\frac{d\tilde{x}}{d\tilde{t}}$ has to have units as $m\tilde{g}$

$[m\tilde{g}] = MLT^{-2}$

$\left[k\frac{d\tilde{x}}{d\tilde{t}}\right] = MLT^{-2}$ $\quad\therefore$

$[k]\frac{L}{T} = MLT^{-2}$

solving for $[k] = [MT^{-1}]$

Let's ignore $k\, d\tilde{x}/d\tilde{t}$ for now:

When $\dfrac{\hat{x}}{R}$ is small, we require that

V remain suitably small.

So what is the typical situation?

set $k=0$ $\quad \dfrac{d^2\hat{x}}{d\hat{t}^2} = -\tilde{g}$

integrating twice:

$$\hat{x} = -\frac{1}{2}\tilde{g}\hat{t}^2 + V\hat{t} + 0.$$

If projectile is hurled upward, it reaches

maximum height when $\dfrac{d\tilde{x}}{d\tilde{t}} = 0$ or

$\tilde{t}_{max} = V/\tilde{g}$

$[\hat{t}] = [T]$ $\qquad t = \hat{t}/(V/\tilde{g})$

and length as $x = \tilde{x}/(V^2/\tilde{g})$

Now, suppose zero drag:

$k=0$ $\therefore$ $\frac{1}{(V/\tilde{g})}\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\tilde{g}$

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\tilde{g}\left(\frac{V}{\tilde{g}}\right)^2 = -V^2/\tilde{g}$$

if $V^2/\tilde{g} \sim O(1)$

then the scale

$$x = \frac{\hat{x}}{V^2/g}$$ leads to

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = -\tilde{g} \Rightarrow \frac{d^2x}{dt^2}\frac{V^2}{\tilde{g}}\frac{1}{(V/\tilde{g})^2} = -g$$

or $$\frac{d^2x}{dt^2} = -1$$

Let's bring in k: recall that

$[k] = [MT^{-1}]$

$$\beta \equiv \frac{k}{m/(V/g)}$$ has unit dimensions

$$\therefore \quad k\frac{d\tilde{x}}{d\tilde{t}} = \beta \frac{dx}{dt}$$, scaling everything

$$\left[\begin{array}{l} \dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} = -1 \qquad \text{(ODE)} \\ \left.\begin{array}{l} x(0) = 0 \\ \dfrac{dx}{dt}(0) = 1 \end{array}\right\} \text{(IC)} \end{array}\right.$$

SOLUTION: let $v = \dfrac{dx}{dt}$ then ODE is

$$(\ddagger) \quad \frac{dv}{dt} + \beta v = -1$$ a linear first order ODE

Multiply $(\ddagger)$ by integrating factor

$$I = e^{\beta t}$$

$$I\frac{dv}{dt} + \beta I v = -I$$

$$\frac{d}{dt}(Iv) = -I$$

integrate both sides

$$e^{\beta t} v = -\int_0^t e^{\beta s}\, ds + c$$

$$v = -e^{-\beta t}\int_0^t e^{\beta s}\, ds + ce^{-\beta t}$$

$$v = -\frac{e^{-\beta t}}{\beta}(e^{\beta t} - 1) + ce^{-\beta t}$$

Apply I.C.

$$v(0) = 1 \Rightarrow c = 1$$

$$v(t) = \left(\frac{1}{\beta} + 1\right)e^{-\beta t} - \frac{1}{\beta} \quad (\star)$$

since $\frac{dx}{dt} = v \Rightarrow$ integrate $(\star)$ to get $x(t)$:

Apply $x(0)=0$ to get

$$x = -\frac{t}{\beta} - \frac{1}{\beta^2}(1+\beta)\left(e^{-\beta t}-1\right)$$

In summary:

$$(\ast)\quad \begin{cases} x(t) = -\dfrac{t}{\beta} + \dfrac{1}{\beta^2}\left(1-e^{-\beta t}\right)(1+\beta) \\ v(t) = \left(\dfrac{1}{\beta}+1\right)e^{-\beta t} - \dfrac{1}{\beta} \end{cases}$$

When $\beta \to 0$, we expect $(\ast)$ to look like the solution of

$$\begin{cases} \dfrac{d^2x}{dt^2} + 1 = 0 \\ x(0)=0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2}t^2 + t \\ v = -t+1 \end{cases}$$

Let $\beta \ll 1$ $\quad e^{-\beta t} \approx 1 - \beta t + \frac{1}{2}\beta^2 t^2, \quad (\ast\ast)$

for small βt.

$$v(t) = \frac{1}{\beta}e^{-\beta t} + e^{-\beta t} - \frac{1}{\beta} \approx \frac{1}{\beta}\left(1 - \beta t + \frac{1}{2}\beta^2 t^2 + \dots\right) + 1 - \beta t + \frac{1}{2}\beta^2 t^2 + \dots - \frac{1}{\beta}$$

Having used the (**) expansion

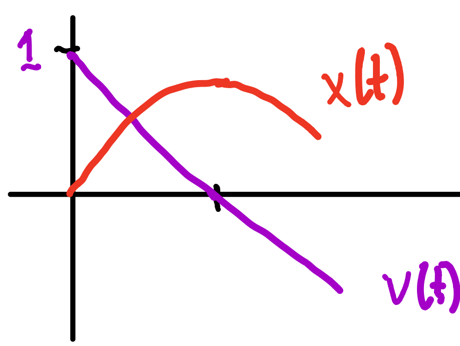
$$v(t) \approx 1 - t + O(\beta)$$

So qualitatively, the limit checks out

$$x(t) \approx -\frac{t}{\beta} + \frac{1+\beta}{\beta^2} - \frac{1}{\beta^2}e^{-\beta t}(1+\beta)$$

$$= -\frac{t}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta} - \frac{1}{\beta^2}\left(1 - \beta t + \frac{1}{2}\beta^2 t^2 + \dots\right) - \frac{1}{\beta}\left(1 - \beta t + \frac{1}{2}\beta^2 t^2 + \dots\right)$$

$$x(t) \approx t - \frac{1}{2}t^2 - \frac{1}{2}\beta t^2 + O(\beta^2) \approx t - \frac{1}{2}t^2 + O(\beta)$$



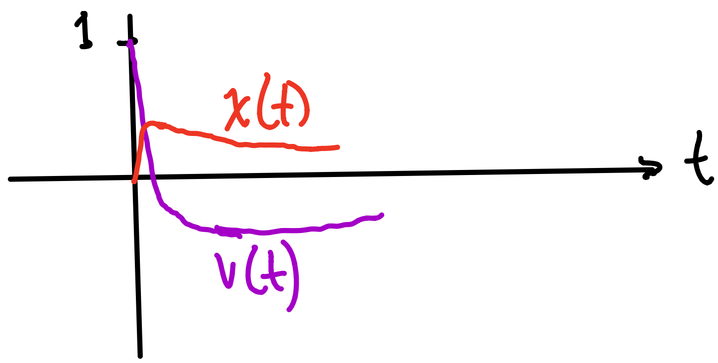
$\beta << 1$

"Gravity Dominates"

What about $\beta >> 1$ "Drag Dominates"

$$\begin{cases} v(t) \approx e^{-\beta t} \\ x(t) = -\frac{t}{\beta} + \frac{1}{\beta^2}(1+\beta)\left(1-e^{-\beta t}\right) \approx -\frac{t}{\beta} + \frac{1}{\beta}\left(1-e^{-\beta t}\right) \end{cases}$$

$$\approx \frac{1}{\beta}\left(1-e^{-\beta t}\right)$$



Rmk: Download and run Mathematica Notebook.