GENERAL STRATEGY
FOR SOLVING SYSTEMS OF FIRST ORDER ODE WITH CONSTANT (MATRIX) EQUATIONS

We will focus on solving the system of n first order constant coefficient equations of the form

$$(IVP)\begin{cases} \dot{\underset{\sim}{x}} = A\underset{\sim}{x} & t>0 \quad (ODE) \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 & (IC) \end{cases}$$

here, $\dot{\underset{\sim}{x}} = \dfrac{d\underset{\sim}{x}}{dt}$

A is $n \times n$ (constant) matrix, $A \in \mathbb{C}^{n\times n}$

$\underset{\sim}{x}(t)$ is an $n \times 1$ (vector), $\underset{\sim}{x}(t) \in \mathbb{C}^n$

$\underset{\sim}{x}_0$ is a constant vector, $\underset{\sim}{x}_0 \in \mathbb{C}^n$

The General Solution to ODE: $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

$$\underset{\sim}{x}(t) = \underset{\sim}{x}_1(t) + \underset{\sim}{x}_2(t) + \dots \underset{\sim}{x}_n(t)$$

What is the form of $\underset{\sim}{x}_i(t)$??

Assume a solution of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ of the form

$$(*) \quad \underset{\sim}{x} = c e^{\lambda t} \underset{\sim}{v}$$

$c \in \mathbb{C}$, $\lambda \in \mathbb{C}$, $\underset{\sim}{v} \in \mathbb{C}^n$

Substitute $(*)$ into $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ and obtain

$$\lambda \cancel{c e^{\lambda t}} \underset{\sim}{v} = A \cancel{c e^{\lambda t}} \underset{\sim}{v} \quad \text{, or}$$

$$(A - \lambda I)\underset{\sim}{v} e^{\lambda t} = 0 \qquad e^{\lambda t} \neq 0$$

$$\therefore \quad (A - \lambda I)\underset{\sim}{v} = 0$$

This is an eigenvalue problem, $\underset{\sim}{v}$ are the eigenvectors, λ the corresponding eigenvalues.

<u>Rmk:</u> $\underset{\sim}{v} = \emptyset$ is always a solution, but now we are seeking $\underset{\sim}{v} \neq \emptyset$ solutions.

For $\underset{\sim}{v} \neq \emptyset$ we require that

$$\det(A - \lambda I) = 0$$

which yields the eigenvalues λ_i. Once these are found, we proceed to find eigenvectors, via $(A - \lambda_i I)v = 0$

Rmk: There may be up to n unique eigenvalues.
To each eigenvalue there might one or more eigenvectors

ex) Find all solutions to

$$\dot{\underset{\sim}{x}} = \frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} \quad \text{where}$$

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix},$$ Find roots of

characteristic equation:

$$p(\lambda) = \det(A - I\lambda) = 0$$

$$= \det\begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & 1-\lambda \end{pmatrix} = 0$$

$$p(\lambda) = (1-\lambda)(\lambda-3)(\lambda+2) = 0$$ cubic equation

$$\therefore \lambda_1 = 1, \lambda_2 = 3, \quad \lambda_3 = -2$$

all simple eigenvalues, all unique.

Find eigenvectors associated with each e'value:

(i) for $\lambda_1 = 1$

$$(A - I)\underset{\sim}{v}_1 = \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = 0$$

a suitable $\underset{\sim}{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

(ii) for $\lambda_2 = 3$

$$(A - 3I)\underset{\sim}{v}_2 = \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & +1 & -4 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = 0$$

$$\underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

(iii) for $\lambda_3 = -2$ $\quad (A+2I)\underset{\sim}{v}_3 = \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix} = 0 \Rightarrow \underset{\sim}{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ The general solution

$$\underset{\sim}{x} = c_1 \underset{\sim}{v}_1 e^{\lambda_1 t} + c_2 \underset{\sim}{v}_2 e^{\lambda_2 t} + c_3 \underset{\sim}{v}_3 e^{\lambda_3 t}$$

$$\underset{\sim}{x} = c_1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t}$$

or $$\underset{\sim}{x} = \begin{pmatrix} -c_1 e^{t} + c_2 e^{3t} - c_3 e^{-2t} \\ 4c_1 e^{t} - 2c_2 e^{3t} + c_3 e^{-2t} \\ c_1 e^{t} + c_2 e^{3t} + c_3 e^{-2t} \end{pmatrix}$$ //

Rule: In this case we got unique eigenvalues, n of them. We next deal with 2 issues:

* COMPLEX EIGENVALUES: this is a case where we can cast solutions in terms of trigonometric or hyperbolic functions. So, this is not a case of non-distinct eigenvalues, but rather, that for complex eigenvalues which come as

conjugates, we can rewrite these in simpler ways.

* **REPEATED ROOTS**: in this case we might expect a repeated root might have more than 1 eigenvector associated with it. We need a procedure for finding these.

COMPLEX ROOTS

To each $\lambda = \alpha + i\beta$ root of $p(\lambda) = 0$, there will be $\bar{\lambda} = \alpha - i\beta$

They come as complex conjugate pairs.

Here we describe a way to express the eigenvectors in a useful way:

EULER'S IDENTITIES

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Suppose $p(\lambda)=0$ generates one or more complex conjugate pairs. For each pair $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \bar{\lambda}_1 = \alpha - i\beta$ the solution $\underset{\sim}{x}(t)$ will contain the terms

$$c_1 e^{\lambda_1 t} \underset{\sim}{v}_1 + c_2 e^{\lambda_2 t} \underset{\sim}{v}_2$$

$$c_1 e^{(\alpha+i\beta)t} \underset{\sim}{v}_1 + c_2 e^{(\alpha-i\beta)t} \underset{\sim}{v}_2$$

$$\text{or } e^{\alpha t}\left[c_1 e^{i\beta t} \underset{\sim}{v}_1 + c_2 e^{-i\beta t} \underset{\sim}{v}_2\right]$$

We could leave these like that, or we could come up with an alternative representation. In what follows we detail this alternative:

Take both: $\lambda = \alpha \pm i\beta$ (these are actually two roots, two eigenvectors)

then $$e^{\alpha t}(\cos\beta t \pm i\sin\beta t)(\underset{\sim}{v} \pm i\underset{\sim}{w})$$

where $\underset{\sim}{v}$ & $\underset{\sim}{w}$ are the eigenvectors associated with $\lambda = \alpha \pm i\beta$ (i.e. $\det(A-\lambda I)=0$)

Rmk: the arbitrary constants c_1 & c_2 have been absorbed into the definition of $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

We note that $(\underset{\sim}{v}\cos\beta t - \underset{\sim}{w}\sin\beta t)e^{\alpha t}$

and $(\underset{\sim}{v}\sin\beta t + \underset{\sim}{w}\cos\beta t)e^{\alpha t}$

are linearly independent solutions to

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x}.$$

So we use these as the 2 vectors associated with λ and $\bar{\lambda}$.

ex) Solve

IVP $$\begin{cases} \dot{\underset{\sim}{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \underset{\sim}{x}, \quad t>0 & \text{(ODE)} \\ \underset{\sim}{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \text{(IC)} \end{cases}$$

We assume $\underset{\sim}{x} = ce^{\lambda t}\underset{\sim}{v}$ then we get

$$(A - \lambda I)\underset{\sim}{v} = 0$$

$$p(\lambda) = \det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(\lambda^2 - 2\lambda + 2) = 0$$

with roots $\begin{cases} \lambda_1 = 1 \\ \lambda_{2,3} = 1 \pm i \end{cases}$

for $\lambda_1 = 1$

$$(A - I)\underset{\sim}{v_1} = 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\underset{\sim}{v_1} = 0 \Rightarrow \underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \underset{\sim}{x_1}(t) = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 1+i$ (We can do both $\lambda_2 = 1+i$, $\lambda_3 = 1-i$):

$$(A - (1+i)I)\underset{\sim}{v_2} = 0 = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 1 & -i \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = 0$$

pick $\underset{\sim}{v_2} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$$\therefore \underset{\sim}{x_2}(t) = e^{(1+i)t}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = (\cos t + i\sin t)e^t\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

$$= (\cos t + i\sin t)\left[\begin{pmatrix}0\\ i\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right]e^t$$

$$= e^t\begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix} + i e^t\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}$$

these 2 vectors are linearly independent so use these to define $\underset{\sim}{x_2}$ & $\underset{\sim}{x_3}$:

$$(¥)\quad \underset{\sim}{x}(t) = c_1\underset{\sim}{x_1}(t) + c_2 e^t\begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix} + c_3 e^t\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}$$

Note: we absorbed the constant i into the definition of c_3

Next, we apply (IC) to (¥)

set $t=0$ in (¥)

$$c_1\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + c_2\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + c_3\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} = \underset{\sim}{x_0} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

$\therefore\ c_1 = c_2 = c_3 = 1$

The final solution:

$$\underset{\sim}{x}(t) = \left[\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix} + \begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}\right]e^t$$

$$= e^t\begin{pmatrix}1\\ -\sin t + \cos t\\ \cos t + \sin t\end{pmatrix}$$

Solving Systems of ODE:

$$\text{IVP}\begin{cases} \text{ODE} & \dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{F} \quad t>0 \\ \text{I.C.} & \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \end{cases}$$

$$\underset{\sim}{x}(t) \in \mathbb{R}^n \qquad A \in \mathbb{R}^{n\times n}$$

$$\underset{\sim}{F} = \underset{\sim}{F}(t) \in \mathbb{R}^n \quad , \quad \underset{\sim}{x}_0 \in \mathbb{R}^n$$

where $\dot{(\ )} \equiv \frac{d}{dt}(\ )$

The general solution to the IVP is

$$\underset{\sim}{x}(t) = \underbrace{\underset{\sim}{x}_H(t)}_{\text{Homogeneous}} + \underbrace{\underset{\sim}{x}_p(t)}_{\text{particular}}$$

where $\dot{\underset{\sim}{x}}_H = A\underset{\sim}{x}_H$

$$\dot{\underset{\sim}{x}}_p = A\underset{\sim}{x}_p + \underset{\sim}{F}$$

THE REPEATED ROOT CASE

ex) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ (*)

cast this as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 - 2x_2 \end{cases}$$

or $\dot{\underset{\sim}{x}} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \underset{\sim}{x}$ $\quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t)$

Let's solve (*) : $y_i(t) = e^{\alpha_i t}$ $\quad i = 1, 2$

substitute into (*)

$$\alpha^2 + 2\alpha + 1 = 0$$

$$(\alpha + 1)^2 = 0$$

$\therefore$ has a repeated root $\alpha = -1$

let $y_1 = e^{-t}$ is a solution

To find a second one "Reduction of order"

technique:

let $y_2 = te^{-t} = t y_1$

the general solution

$$y = Ae^{-t} + Bte^{-t}$$

where A, B are constant parameters. //

Let $\underset{\sim}{x}_i = \begin{pmatrix} y_i \\ \frac{dy_i}{dt} \end{pmatrix}$

then $\underset{\sim}{x}_1 = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} x_A^1 \\ x_B^1 \end{pmatrix}$

$$\underset{\sim}{x}_2 = \begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix} = \begin{pmatrix} x_A^2 \\ x_B^2 \end{pmatrix}$$

$$\frac{d}{dt}\underset{\sim}{x}_1 = \frac{d}{dt}\begin{pmatrix} x_A^1 \\ x_B^1 \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_A^1 \\ x_B^1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1(-e^{-t}) \\ -1(-e^{-t}) & -2e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} \checkmark$$

Also, note that

$$\begin{pmatrix} (1-t)e^{-t} \\ (t-2)e^{-t} \end{pmatrix} = \frac{d\underset{\sim}{x}_2}{dt} = A\underset{\sim}{x}_2 = \begin{pmatrix} (1-t)e^{-t} \\ -te^{-t} - 2(1-t)e^{-t} \end{pmatrix} \checkmark$$

Applying I.C:

Suppose $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then

$$\underset{\sim}{x}(t) = c_1\underset{\sim}{x}_1 + c_2\underset{\sim}{x}_2 = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} (c_1 + c_2 t)e^{-t} \\ (c_2 - c_1 - c_2 t)e^{-t} \end{pmatrix}$$

Apply I.C. $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 - c_1 \end{pmatrix}$

$\therefore \quad c_1 = 1 \qquad c_2 = 1 + c_1 = 2$

$$\underset{\sim}{x}(t) = \begin{pmatrix} (1+2t)e^{-t} \\ (1-2t)e^{-t} \end{pmatrix} \quad //$$

<u>Rmk:</u> A note on applying initial conditions. Suppose we are solving $\underset{\sim}{\dot{x}} = A\underset{\sim}{x} + \underset{\sim}{f}$, $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$.

The solution is $\underset{\sim}{x}(t) = \underset{\sim}{x}_H(t) + x_p(t)$. $\underset{\sim}{x}_H(t)$ will have the form

$$\underset{\sim}{x}_H = c_1 \underset{\sim}{x}_1(t) + c_2 \underset{\sim}{x}_2(t) + \dots c_n \underset{\sim}{x}_n(t)$$

The c_i's are determined by the initial conditions.

Apply I.C. as the very last step in the process, after you know x_H and x_p. That is,

$$\underset{\sim}{x}(0) = x_H(0) + x_p(0)$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0 = c_1 x_1(0) + c_2 x_2(0) \dots + c_n x_n(0) + x_p(0)$$

//

Equal Roots Case:

How do we find n fundamental solutions to the n-dimensional problem, when there are repeated eigenvalues?

Take $\quad \dot{\underset{\sim}{x}} = A\underset{\sim}{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underset{\sim}{x}$

here A has only 2 distinct e'values: 1, 2.

so $\quad \underset{\sim}{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} \qquad \underset{\sim}{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$

how to find $\underset{\sim}{x}_3$? use "Reduction of order"

<u>Aside:</u> The matrix exponential

Given e^{At} $\qquad A \in \mathbb{R}^{n \times n}, t \in \mathbb{R}$

e^{At} is a matrix

Recall $\quad e^{\alpha t} = \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!}$ for scalar exponential

Also

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots$$

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \quad (\ast)$$

Taking the derivative of

$$\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + \frac{1}{2}A^2t^2 + \cdots\right)$$

$$= A + A^2 t + \cdots + \frac{A^{n+1}t^n}{n!} + \cdots$$

$$= A\left(I + At + \cdots \frac{A^n t^n}{n!} + \cdots\right) = Ae^{At}$$

The idea is to find n linearly independent e'vectors $\underset{\sim}{v}$'s (for the n fundamental solutions) for which the infinite series of $e^{At}v$ can be summed.

<u>Fact:</u> $e^{At}\underset{\sim}{v} = e^{(A-\lambda I)t}e^{\lambda It}\underset{\sim}{v}$

for any constant λ, i.e.

$(A-\lambda I)\lambda I = \lambda I(A-\lambda I)$

commute

Also $e^{\lambda It}\underset{\sim}{v} = [I + \lambda It + \frac{1}{2}\lambda^2t^2I^2 + \cdots]\underset{\sim}{v}$

$= [1 + \lambda t + \frac{1}{2}\lambda^2t^2 + \cdots]\underset{\sim}{v} = e^{\lambda t}\underset{\sim}{v}$

$\therefore\ e^{At}\underset{\sim}{v} = e^{\lambda t}e^{(A-\lambda I)t}\underset{\sim}{v}$ //

The crux of the following calculation is the fact that if $\underset{\sim}{v}$ satisfies $(A-\lambda I)^m\underset{\sim}{v} = 0$ for some $m \in \mathbb{Z}$ (integer) then the series

$e^{(A-\lambda I)t}\underset{\sim}{v}$ terminates at the m^{th} term:

That is, if $(A-\lambda I)^m\underset{\sim}{v} = 0$ then so will

$$(A-\lambda I)^m (A-\lambda I)^\ell \underset{\sim}{v} = 0,$$
for any positive ℓ.

$$\therefore e^{(A-\lambda I)t}\underset{\sim}{v} = \underset{\sim}{v} + t(A-\lambda I)\underset{\sim}{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\underset{\sim}{v},$$ a finite series.

$$\therefore e^{At}\underset{\sim}{v} = e^{\lambda t}\left[\underset{\sim}{v} + (A-\lambda I)\underset{\sim}{v} + \cdots + \frac{t^{m-1}}{(m-1)!}(A-\lambda I)^{m-1}\underset{\sim}{v}\right].$$

<u>exactly</u>.

ALGORITHM: FINDING n FUNDAMENTAL SOLUTIONS TO $\dot{x} = Ax$, where $A \in \mathbb{R}^{n\times n}$:

① If $\dot{x} = Ax$ has n simple eigenvalues, then find the n e'vectors.

<u>Rmk:</u> $e^{(A-\lambda_i I)t}\underset{\sim}{v}$ terminates as a series.

①' If there are repeated e'values, some of these will permit finding the additional e'vectors by inspection.

(2) Suppose A has $k < n$ (linear independent) eigenvectors.
For these first k, use the procedure in (1) & (1'):

for $\dot{\underline{x}} = A\underline{x}$ the solutions for these are of the form $e^{\lambda_i t}\underline{v}_i$ $i = 1, 2, .., k$.

To find the $k+1, k+2, .., n$ remaining solutions to $\dot{x} = Ax$, use reduction of order: Suppose λ is double root. Then the second eigenvector $\underline{v}$ should be chosen so that $(A - \lambda I)\underline{v} \neq 0$ but $(A - \lambda I)^2 \underline{v} = 0$. Suppose, triply repeated. In that case choose the third eigenvector so that $(A - \lambda I)v \neq 0$, $(A - \lambda I)^2 v \neq 0$ and $(A - \lambda I)^3 v = 0$... and so on.

The remaining part of the reduction of order algorithm is described by the following example:

ex) Find general solution to

$\dot{\underset{\sim}{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underset{\sim}{x}$. The general solution is

$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$

$p(\lambda) = (1-\lambda)^2 (2-\lambda) = 0$

$\lambda = 1$, repeated

$\lambda = 2$ is simple.

for $\lambda = 2$ $\quad (A - 2I)\underset{\sim}{v} = 0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$$\underset{\sim}{x}_3 = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 1$: $\quad (A - I)\underset{\sim}{v} = 0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$\underset{\sim}{x}_1 = e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ so this is the usual thing.

The reduction of order process will find the second solution associated with $\lambda = 1$:

Want $(A - I)\underset{\sim}{v} \neq 0$ but $(A - I)^2 \underset{\sim}{v} = 0$

$$(A-I)^2 v = 0 = \begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = 0$$

$\underline{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$. Note that $\begin{pmatrix}0\\1\\0\end{pmatrix}$ does not satisfy

$$(A-I)\begin{pmatrix}0\\1\\0\end{pmatrix} = 0.$$

$$x_2(t) = e^{At}\begin{pmatrix}0\\1\\0\end{pmatrix} = e^t e^{(A-I)t}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

note that $\frac{t^2}{2}(A-I)^2 v = 0$ by design. So this is NOT an approximation.

Expanding

$$\underline{x}_2(t) = e^t\left[I + t(A-I)\right]\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= e^t\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]$$

$$= e^t\begin{pmatrix}0\\1\\0\end{pmatrix} + te^t\begin{pmatrix}1\\0\\0\end{pmatrix} = e^t\begin{pmatrix}t\\1\\0\end{pmatrix}$$

So now we have all 3 expected fundamental solutions. The general solution is

$$\underset{\sim}{x} = c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + c_3 \underset{\sim}{x}_3$$

$$= c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} (c_1 + c_2 t) e^t \\ c_2 e^t \\ c_3 e^{2t} \end{pmatrix}$$

NON HOMOGENEOUS PROBLEM

$$\text{IVP} \begin{cases} \text{ODE} & \dot{\underset{\sim}{x}} = A \underset{\sim}{x} + \underset{\sim}{f} \\ \text{IC} & \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \end{cases}$$

$A \in \mathbb{R}^{n \times n}$

$\underset{\sim}{x}(t) \in \mathbb{R}^n$

$\underset{\sim}{x}_0 \in \mathbb{R}^n$

$\underset{\sim}{f}(t) \in \mathbb{R}^n$

The IVP solution is

$$x(t) = \underset{\sim}{x}_H + \underset{\sim}{x}_P$$

$$\text{where} \begin{cases} \dot{\underset{\sim}{x}}_H = A \underset{\sim}{x}_H & (1) \\ \dot{\underset{\sim}{x}}_p = A \underset{\sim}{x}_p + \underset{\sim}{f} & (2) \end{cases}$$

Solve (1) $\underset{\sim}{x}_H = c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + \cdots + c_n \underset{\sim}{x}_n$

Which can be written as

$$\underset{\sim}{x}_H = X \underset{\sim}{c}$$

$$\underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$X = \begin{pmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \end{pmatrix}$$

$n \times n$ matrix call "Fundamental solution Matrix"

ex) $\underset{\sim}{x} = c_1 e^{t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

solves $\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \underset{\sim}{x}$.

Let $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$X = \begin{pmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ 4e^{t} & 2e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{pmatrix}$$, the solution can be expressed as $\mathbf{x} = X(t)\,\mathbf{c}$

<u>Aside remark:</u>

For $\dot{\mathbf{x}} = A\mathbf{x}$ we can write

$$e^{At} = X(t)\,X^{-1}(t=0)$$

$$e^{At} = X(t)\,X^{-1}(0)$$

Since $t=0$ $e^{A0} = I = X(0)\,X^{-1}(0)$.

Also, $\dot{x}_i = A x_i \qquad i = 1, 2, \ldots, n$

or $\dfrac{dX}{dt} = A X$ is the same thing.

$\therefore$ e^{At} is a solution to $X = AX$ $(*)$

$\Rightarrow$ e^{At} and X are related to each via $(*)$

To find $x_p(t)$:

$(\star)$ $\dot{x}_p = A x_p + f$

Use variation of parameters:

$(\$)$ $x_p(t) = X(t)\, u(t)$

$u(t) \in \mathbb{R}^n$ of unknown functions

if $(\$)$ is a solution to $(\star)$

$$\dot{x}_p = \cancel{\dot{X} u} + X \dot{u} = \cancel{A X u} + f$$

$$\therefore \quad X \dot{u} = f$$

or $\dot{\underset{\sim}{u}} = \underline{X}^{-1} \underset{\sim}{f}$ computing antiderivative:

$$\therefore \underset{\sim}{u}(t) = \int_0^t \underline{X}^{-1}(s)\, \underset{\sim}{f}(s)\, ds + \underset{\sim}{u}(0)$$

$\underset{\sim}{u}(0)$ gets absorbed into initial conditions.

$$\therefore \underset{\sim}{x}(t) = \underline{X}(t)\, \underline{X}^{-1}(0)\, \underset{\sim}{x}_0 + \underline{X} \int_0^t \underline{X}^{-1}(s)\, \underset{\sim}{f}(s)\, ds$$

where $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$

ex) $\dot{\underset{\sim}{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \underset{\sim}{x} + \begin{pmatrix} 0 \\ 0 \\ e^t \cos 2t \end{pmatrix} = A \underset{\sim}{x} + \underset{\sim}{f}$, $t > 0$

$x(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\underset{\sim}{x} = \underset{\sim}{x}_H + \underset{\sim}{x}_p$$

$$\dot{\underset{\sim}{x}}_H = A \underset{\sim}{x}_H$$

$$p(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda_1 = 1 \qquad \lambda_{2,3} = 1 \pm 2i$$

for λ_1 $\quad \underset{\sim}{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \qquad \underset{\sim}{x}_1 = e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

$$\lambda_{2,3} = 1 \pm 2i$$

$$\underset{\sim}{v} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \quad \therefore x(t) = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} e^{(1+2i)t}$$

$$= e^t (\cos 2t + i \sin 2t) \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -i \end{pmatrix} \right)$$

$$= e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + i e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

$$\therefore \underset{\sim}{x}_2(t) = e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \qquad \underset{\sim}{x}_3(t) = e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

$$\mathbb{X} = \begin{pmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t \sin 2t \\ 2e^t & e^t \sin 2t & -e^t \cos 2t \end{pmatrix}$$

$$\mathbb{X}(0) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\mathbb{X}^{-1}(0) = \begin{pmatrix} 1/2 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$e^{At} = \mathbb{X}(t)\,\mathbb{X}^{-1}(0)$$

$$e^{At} = e^t \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} + \frac{3}{2}\cos 2t + \sin 2t & \cos 2t & -\sin 2t \\ 1 + \frac{3}{2}\sin 2t - \cos 2t & \sin 2t & \cos 2t \end{bmatrix}$$

$$\underset{\sim}{x}(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mathbb{X}\int_0^t \underbrace{e^{-s}\,F(s)}_{\mathbb{X}^{-1}} \begin{pmatrix} 0 \\ 0 \\ e^s \cos 2s \end{pmatrix} ds$$

$$F(t) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} + \frac{3}{2}\cos 2t + \sin 2t & \cos 2t & -\sin 2t \\ 1 + \frac{3}{2}\sin 2t - \cos 2t & \sin 2t & \cos 2t \end{bmatrix}^{-1}$$

$$\underset{\sim}{x}(t) = e^{t}\begin{pmatrix} 0 \\ \cos 2t - \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix} + \mathbb{X} \int_0^t \begin{bmatrix} 0 \\ \sin 2s \cos 2s \\ \cos^2 2s \end{bmatrix} ds$$

$$= e^{t} \begin{bmatrix} 0 \\ \cos 2t - (1 + \frac{1}{2}t)\sin 2t \\ (1 + \frac{1}{2}t)\cos 2t + \frac{5}{4}\sin 2t \end{bmatrix}$$

//

ex) Solve $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{f}$ $\quad t > 0$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \underset{\sim}{f} = \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = 0 \quad \therefore \quad \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 3 \end{matrix}$$

$$\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \quad \underset{\sim}{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \qquad \underset{\sim}{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$\underline{X} = \begin{pmatrix} e^{t} & e^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

$$\underline{X}(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \therefore \quad \underline{X}^{-1}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{x}(t) = \underline{X}\,\underset{\sim}{c} + \underline{X}\int_0^t \underline{X}^{-1} f \, ds = \underline{X}\,\underset{\sim}{c} + \underset{\sim}{x}_p$$

<u>Rule</u>: To find inverse of 2x2 matrix:

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det X = e^{4t}$$

$$X^{-1} = e^{-4t}\begin{bmatrix} e^{3t} & -e^{-3t} \\ 0 & e^{t} \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-7t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$X^{-1}\underset{\sim}{f} = \begin{bmatrix} e^{-t} & -e^{-7t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ \bullet \end{bmatrix} \begin{bmatrix} -e^{-7t}\bullet \\ e^{-2t}\bullet \end{bmatrix}$$

$$X^{-1}f = \begin{bmatrix} -e^{-7t}\, t e^{2t} \\ e^{-2t}\, t\, e^{2t} \end{bmatrix} = \begin{bmatrix} te^{-5t} \\ t \end{bmatrix}$$

$$\int_0^t X^{-1} f\, ds = \int_0^t \begin{bmatrix} se^{-5s} \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} -e^{-5t}\left(\frac{1}{25} + \frac{t}{5}\right) \\ \frac{1}{2}t^2 \end{bmatrix}$$

$$\underset{\sim}{x}_p = \underset{\sim}{\Phi} \begin{bmatrix} -\frac{1}{5}e^{-5t}\left(\frac{1}{5}+t\right) \\ \frac{1}{2}t^2 \end{bmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5}e^{-5t}e^{t}\left(t+\frac{1}{5}\right) + \frac{1}{2}t^2 e^{3t} \\ \frac{1}{2}t^2 e^{3t} \end{pmatrix}$$

and $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\sim}{\Phi}(0)\,\underset{\sim}{c}$

or $\underset{\sim}{\Phi}^{-1}(0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \therefore c_1 = 1, \; c_2 = 0$$

$$\therefore\ \underset{\sim}{x}(t) = \begin{pmatrix} e^{t} - \frac{1}{5}e^{-5t}e^{t}\left(t+\frac{1}{5}\right) + \frac{1}{2}t^{2}e^{3t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$