SYSTEMS OF 1st ORDER ODES
(Constant Matrix Case)

BACKGROUND:

Classification of Equations

Algebraic Equations

$$g(x, y, \dots) = 0$$

explicitly linear or nonlinear algebraic equations.

DIFFERENTIAL EQS: n^{th} order diff eq

ODE: $$G(y(x), x, y'(x), y''(x), \dots y^{(n)}(x)) = 0$$

$y(x)$ ← dependent variable; x ← independent variable

PDE: Partial Diff Eq:

$$H(u(x,y,z\dots), u_x, u_y, u_z \dots, u_{xx}, u_{xy}, u_{zz}, \dots) = 0$$

u ← dependent

INTEGRAL EQUATIONS

integrals appear

e.g. $$y(t) + \int_0^t K(t-\tau)\,y(\tau)\,d\tau = 0 \quad (*)$$

Solve for $y(t)$, $K(t)$ is given

Differential/Integral Equations

$$\frac{dy}{dt} + g(t)\,y = \int_0^t h(t-\tau)\,y(\tau)\,d\tau$$

Solving for y, know $g(t)$, $h(t)$

Major Classification of All Equations:

- linear (in the dependent variable(s))
- nonlinear (in the dependent variable(s))

ex) $y'' + \sin y = 0 \qquad y = y(x)$

is a 2nd order ODE, nonlinear

(because $\sin y$ is not linear function)

$y'' + \omega^2 y = 0 \qquad y = y(x)$

ω^2: constant

linear

2nd order ODE

ex) Nonlinear pde

$$u_t + u u_x - \beta u_{xxx} = 0$$

KdV $\quad u = u(x,t)$

has "solitary-wave solutions"

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1st order ODEs

(1) $\frac{dy}{dx} + p(x)\,y(x) = g(x)$

LINEAR

(2) $h(y)\,dy = g(x)\,dx$

SEPARABLE

(3) $M(x,y)\,dy + N(x,y)\,dx = 0$

iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ EXACT

(4) $\frac{dy}{dx} = f(x,y)$ if $f(x,y) = g(y/x)$

homogeneous

let $u = y/x$ then solve for

$y(u,x)\ldots$

Reminder, solve Linear ODE

$\frac{dy}{dx} + p(x)\,y = g(x)$

Trick: multiply b.s. by "integrating factor"

$I(x) = e^{\int p\,dx}$

$$\underbrace{I(x)\frac{dy}{dx} + p(x) I(x)\, y}_{\frac{d}{dx}(I y)} = I(x)\, q(x)$$

$\therefore \frac{d}{dx}(Iy) = I(x)\, q(x)$

integrate b.s.

$$Iy = \int_0^x I(s)\, q(s)\, ds + C$$

$$y = I^{-1}(x) \int_0^x I(s)\, q(s)\, ds + I^{-1}(x)\, C$$

if $y(0)$ is known $\Rightarrow$ C can be determined

ex) $y' + \frac{1}{x} y = \sin x$, $x > \pi$ with $y(\pi) = 1$

$$I = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$x y' + y = x \sin x$$

$$\frac{d}{dx}(xy) = x \sin x$$

$$xy = \int_{\pi}^{x} z \sin z \, dz + c$$

$$y = \frac{1}{x} \int_{\pi}^{x} z \sin z \, dz + \frac{c}{x}$$

integrate by parts

$u = z$ $\quad$ $v = -\cos z$

$du = dz$ $\quad$ $dv = \sin z \, dz$

$$y = \frac{1}{x}\left[-z\cos z \Big|_{\pi}^{x} + \int_{\pi}^{x} \cos z \, dz \right] + \frac{c}{x}$$

$$y = \frac{1}{x}\left[-x\cos x + \pi\cos\pi + \sin z \Big|_{\pi}^{x} \right] + \frac{c}{x}$$

$$y = -\cos x + \frac{\sin x}{x} + \frac{c}{x}$$

$$y(\pi) = 1 \Rightarrow 1 = -\cos\pi + \frac{\sin\pi}{\pi} + \frac{c}{\pi}$$

$$1 = 1 + 0 + \frac{c}{\pi}$$

$$c = 0$$

$$\therefore y(x) = -\cos x + \frac{\sin x}{x}$$ //

Systems of ODE's Linear

Rmk: We focus on 1st order ODE because even high order linear ODE's can be recast as a system of 1st order ODE's.

Take

$$\left(\frac{y}{x}\right) \quad a_n \frac{d^{(n)}y}{dx^n} + a_{n-1}\frac{d^{(n-1)}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

$$a_n \frac{d}{dx} u_{n-1} + a_{n-1}\frac{d}{dx} u_{n-2} \dots + a_1 \frac{d}{dx} u_0 + a_0 u_0 = f(x)$$

Assume $a_n \neq 0$, a_i's are not all zero constants.
We also are supplied with I.C:

$$\left.\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \\ &\vdots \\ y^{(n-1)}(0) &= y_{n-1} \end{aligned}\right\} \text{I.C.}$$

Let $u_0 \equiv y$

$$\frac{du_0}{dx} = u_1$$

$$\frac{du_1}{dx} = u_2$$

$\vdots$ referring to ($\mathcal{J}$)

$$\frac{du_{n-1}}{dx} = -\frac{a_{n-1}}{a_n} u_{n-2} - \frac{a_{n-2}}{a_n} u_{n-3} + \cdots - \frac{a_1}{a_n} u_1 - \frac{a_0}{a_n} u_0 + f(x)/a_n$$

knowing that:

$$a_n \underset{\frac{d}{dx} u_{n-1}}{\underbrace{u_n}} + a_{n-1} u_{n-1} \cdots + a_1 u_1 + a_0 u_0 = f(x)$$

define $\underset{\sim}{U} \equiv \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}(x) \in \mathbb{C}^n$

n-dimensional vector function

$$\frac{d}{dx}\underset{\sim}{U} = A(x)\underset{\sim}{U} + \underset{\sim}{F}(x) \quad \text{"NORMAL FORM"}$$

↑ Matrix

$$\underset{\sim}{F}(x) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ f(x)/a_n \end{bmatrix} \quad \updownarrow n$$

$$A(x) = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} & 0 \end{bmatrix}$$

Rmk: We've converted an n^{th} order ODE into a system of 1st order ODE's of dimension n.

* We need to know to to solve systems of 1st

order ODE's & then we can obtain the solution of n^{th} order ODE.

ex) Convert $\frac{d^3y}{dt^3} + (1+\sin t)\frac{dy}{dt} + 3y = e^t$

$y(0)=1 \quad y'(0)=0 \quad y''(0)=0$

to NORMAL FORM:

let $u_0 = y$

$$\begin{cases} \frac{du_0}{dt} = \frac{dy}{dt} = u_1 \\ \frac{du_1}{dt} = \frac{d^2y}{dt^2} = u_2 \\ \frac{du_2}{dt} = \frac{d^3y}{dt^3} = -(1+\sin t)u_1 - 3u_0 + e^t \end{cases}$$

$\underset{\sim}{U} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$ let $\underset{\sim}{F}(t) = \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$

$a_0 = 3 \quad a_1 = (1+\sin t) \quad a_2 = 1$

$$\frac{d\underset{\sim}{U}}{dt} = A(t)\underset{\sim}{U} + \underset{\sim}{F}(t)$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -(1+\sin t) & 0 \end{bmatrix}$$

since:

$$\begin{cases} \frac{du_0}{dt} = \frac{dy}{dt} = u_1 \\ \frac{du_1}{dt} = \frac{d^2y}{dt^2} = u_2 \\ \frac{du_2}{dt} = \frac{d^3y}{dt^3} = -(1+\sin t)u_1 - 3u_0 + e^t \end{cases}$$

the initial vector $\underset{\sim}{U}(0) = \underset{\sim}{U}_0$

$y(0)=1 \quad y'(0)=0 \quad y''(0)=0$

$$\underset{\sim}{U}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving linear 1st order constant matrix ODE

$$\text{IVP}\begin{cases} \dot{\underset{\sim}{x}} = \frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x} + \underset{\sim}{F} & t>0 \quad \text{ODE} \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 & \text{I.C.} \end{cases}$$

where $\underset{\sim}{x}(t) \in \mathbb{R}^n$

$A \in \mathbb{R}^{n\times n}$

$\underset{\sim}{F}$ & $\underset{\sim}{x}_0 \in \mathbb{R}^n$

$\underset{\sim}{F}(t)$ can be a function of t.

The general solution of ODE is

$$\underset{\sim}{x}(t) = (\underset{\sim}{x}_H + \underset{\sim}{x}_p)(t)$$

where $\frac{d\underset{\sim}{x}_H}{dt} = A\underset{\sim}{x}_H$ & $\frac{d\underset{\sim}{x}_p}{dt} = A\underset{\sim}{x}_p + \underset{\sim}{F}$