SINGULAR PERTURBATION EXPANSIONS

Some failures of the regular perturbation series expansion:

1. When we get secular terms

2. Another type of failure: the perturbed problem is totally different qualitatively so perhaps not even the leading order approximates the exact solution.

ex) BVP

$$\begin{cases} \text{ODE}: & \varepsilon y'' + (1+\varepsilon) y' + y = 0 \quad 0<t<1, \quad |\varepsilon| \ll 1 \\ \text{BV}: & y(0)=0 \quad y(1)=1 \end{cases}$$

A boundary value problem.

assume $y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$

and substitute in BVP

ε^0: $\quad y_0' + y_0 = 0$

$\quad y_0(0) = 0 \quad y_0(1) = 1$

ε^1: $\quad y_1' + y_1 = -y_0'' - y_0'$

$\quad y_1(0) = y_1(1) = 0$

etc

Solving ε^0: $\quad y_0 = ce^{-t}$

if $y(0) = 0 \Rightarrow c = 0$, but can't satisfy $y_0(1) = 1$!

if $y_0 = e^{1-t}$ yields $y_0(1) = 1$ but cannot satisfy $y_0(0) = 0$.

Rmk: the root cause of the failure is not the BVP but the solution strategy... regular series perturbations fail.

Rmk: We note that when $\varepsilon = 0$ in BVP the y'' term drops out & we go from 2nd order ODE to a first order ODE.

The solution would change abruptly as $\varepsilon \to 0$.

This is (usually) a sign that there is singular behavior. We say that the BVP is "singular" because as $\varepsilon \to 0$ the behavior of the BVP changes dramatically.

Inner & Outer Approximations

We can actually solve BVP exactly. We do so, so that we can compare the approximate solutions to the exact one.

The solution to BVP is

$$y(t) = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left[e^{-t} - e^{-t/\varepsilon} \right] \quad (\dagger)$$

$y(t)$ changes rapidly near $t=0$. To see this, we compute the first couple of derivatives of $(\dagger)$

$$y' = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(-e^{-t} + \frac{1}{\varepsilon} e^{-t/\varepsilon} \right)$$

$$y'' = \frac{1}{e^{-1} - e^{-1/\varepsilon}} \left(e^{-t} - \frac{1}{\varepsilon^2} e^{-t/\varepsilon} \right)$$

near $t = 0$ $(\varepsilon \ll 1)$

$$\begin{cases} y'' = O(\varepsilon^{-2}) \\ y' = O(\varepsilon^{-1}) \end{cases}$$

$\therefore$ $\varepsilon y'' = O(\varepsilon^{-1})$ in which BV not small for $0 < t < \varepsilon$

Hence BVP has 2 dominant balances:
$0 < t < \varepsilon$ and $\varepsilon < t < 1$

For $\varepsilon < t < 1$ we approximate BV as

$$\begin{cases} (1+\varepsilon) y' + y \approx 0 \\ y(1) = 1 \end{cases}$$

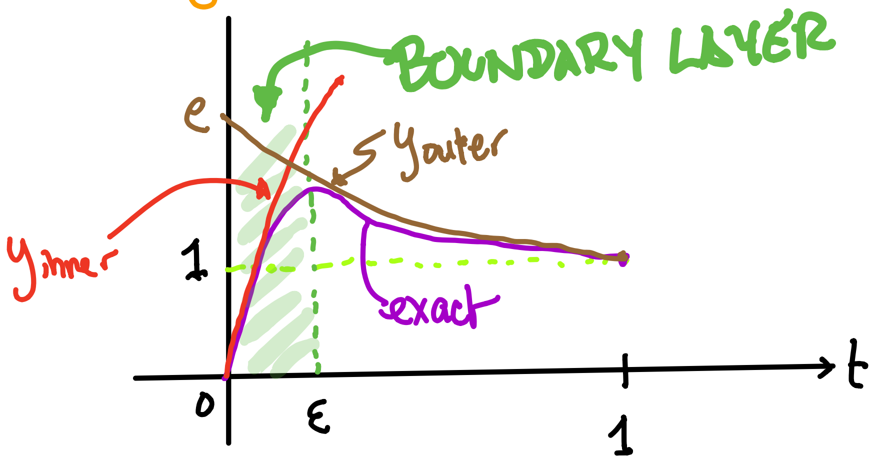
which has a solution $y \equiv y_{outer} = e^{1-t}$

Compare this to (†) exact solution near $t=1$:

$$e^{1-t/\varepsilon} \approx 1$$

For $t \approx 0$ we should get $y_{inner} \approx e - e^{1-t/\varepsilon}$

we will see how later. We need to first learn balancing terms.



Rmk:

Regular Perturbation methods will fail when

① A small parameter multiplies the highest derivative of the equation.

② When setting the parameter to zero completely changes the nature of the equation.

(3) When singular points are present in the domain (i.e. terms in the ODE blow up or become undefined)

(4) When the physical processes underlying the equation has multiple scales (fast/slow, large/small, etc)

<u>Rmk</u>: The BVP is an example of a "boundary layer analysis", wherein we use a singular perturbation method, such as "matched asymptotic expansions"

BALANCING

We will try this first on algebraic equations and then go onto differential equations.

S'pse we want to find the roots of

$$\varepsilon x^2 + 2x + 1 = 0 \qquad 0 < \varepsilon \ll 1$$

Rmk: Clearly, the roots are easy to obtain exactly, but let's proceed by asymptotic balancing

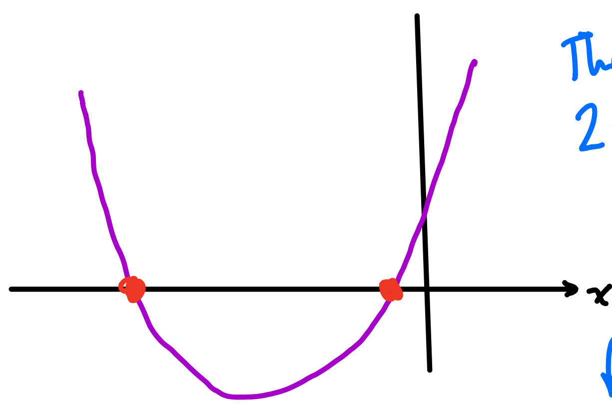
Rmk: Clearly, the $\varepsilon = 0$ case is qualitatively different from the $\varepsilon \neq 0$ case: 1 root instead of 2.

The key to find the 2 roots via an iterative procedure is to figuring out when εx^2 is not small

There are 3 terms, 2 possible balances:

(i) εx^2 & 1 are same order, and $2x \ll 1$.

(ii) εx^2 & $2x$ are same order, and both large compared to 1.



There are 2 negative roots, the more negative root moves far away from $x = 0$.

Case (i) $x = O(1/\sqrt{\varepsilon})$, but
$2x$ can't then be < 1.

Case (ii) $x = O(1/\varepsilon)$ both εx^2 and $2x$ can be made of the same order, both $O(1/\varepsilon)$ and both large compared to 1.

So we dismiss (i) & adopt (ii) scaling

SCALING

Let $\tilde{x} = \dfrac{x}{1/\varepsilon} = O(1)$

Substitute into $\varepsilon x^2 + 2x + 1 = 0$
to get $\tilde{x}^2 + 2\tilde{x} + \varepsilon = 0$ (✱)

Assume
$$\tilde{x} = \tilde{x}_0 + \varepsilon \tilde{x}_1 + \varepsilon^2 \tilde{x}_2 + \cdots$$
and substitute into (✱) and separate the ε orders;

$$\varepsilon^0: \quad \tilde{x}_0^2 + 2\tilde{x}_0 = 0 \Rightarrow \tilde{x}_0(\tilde{x}_0 + 2) = 0$$

so

$$x_0 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\varepsilon^1: \quad 2\tilde{x}_0\tilde{x}_1 + 2\tilde{x}_1 + 1 = 0$$

or $2\begin{pmatrix} 0 \\ -2 \end{pmatrix}\tilde{x}_1 + 2\tilde{x}_1 + 1 = 0$

so $\tilde{x}_1 = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$

so to $O(\varepsilon^2)$

$$\tilde{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \varepsilon\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} + O(\varepsilon^2)$$

reverting back to original variable

$$x = \frac{1}{\varepsilon}\tilde{x} = \begin{cases} -1/2 + \cdots \\ \frac{2}{\varepsilon} + \frac{1}{2} + \cdots \end{cases}$$

How good are these? let's compare to the exact roots:

$$x = -\frac{1}{\varepsilon} \pm \frac{1}{2}\sqrt{\frac{4}{\varepsilon^2} + \frac{4\varepsilon}{\varepsilon^2}} = -\frac{1}{\varepsilon}\left(1 \mp \sqrt{1+\varepsilon}\right)$$

for small ε:

$$x \approx -\frac{1}{\varepsilon}\left(1 \mp \left[1 + \frac{1}{2}\varepsilon + \dots\right]\right)$$

$$x = -\frac{1}{\varepsilon}\begin{cases} \frac{\varepsilon}{2} + \dots \\ 2 + \frac{1}{2}\varepsilon \dots \end{cases} = \begin{cases} \frac{1}{2} + \dots \\ \frac{2}{\varepsilon} + \frac{1}{2} + \dots \end{cases}$$

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BOUNDARY LAYER ANALYSIS

BVP
$$\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0 \quad 0 < t < 1 \quad 0 < \varepsilon \ll 1 \\ \text{(ODE)} \\ y(0) = 0 \quad y(1) = 1 \\ \text{(IC)} \end{cases}$$

Rmk: We already saw that there are two regions in t where the behavior of y is qualitatively different. The "outer region" $\varepsilon < t < 1$ has a solution $y_{outer} = e^{1-t}$

The "inner solution" $0 < t < \varepsilon$ is called the boundary layer solution was stated but not derived. We do that next:

We will rescale t inside the boundary layer so that in this new (scaled) variable, y will change in $\mathcal{O}(1)$.

Let's assume that $\begin{cases} \tau = \dfrac{t}{\delta(\varepsilon)} \\ Y = y \end{cases}$ (¥)

$\delta(\varepsilon) = \varepsilon^p$ typically

Replace the (¥) into BVP

$$(\ni) \quad \frac{\varepsilon}{\delta^2(\varepsilon)} Y''(\tau) + \frac{(1+\varepsilon)}{\delta(\varepsilon)} Y'(\tau) + Y(\tau) = 0$$

$$Y(\tau) = y(\tau\delta(\varepsilon))$$

$$(\cdot)' = \frac{\partial}{\partial\tau}(\cdot)$$

Rmk: There are 4 terms in $(\ni)$ of size

$$\frac{\varepsilon}{\delta^2(\varepsilon)}, \quad \frac{1}{\delta(\varepsilon)}, \quad \frac{\varepsilon}{\delta(\varepsilon)}, \quad 1$$

We consider all possible dominant balances among <u>pairs</u> of terms.

The goal is to find an approximation that captures the dynamics in the Boundary layer, rather than to find an exact balance ∴ we want to keep whatever is important about the exact balance.

So the three balances are:

(i) $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{1}{\delta(\varepsilon)}$ and both ≥ 1 and both $\geq \frac{\varepsilon}{\delta(\varepsilon)}$

(ii) $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{\varepsilon}{\delta(\varepsilon)}$, with $\frac{1}{\delta(\varepsilon)}$ and $\frac{\varepsilon}{\delta(\varepsilon)}$ both small.

(iii) $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{\varepsilon}{\delta(\varepsilon)}$ and $\frac{1}{\delta(\varepsilon)}$ and 1 both small

Cases (ii) and (iii) cannot be achieved $\varepsilon << 1$

In fact in (iii) we require that $\varepsilon \sim 1$, which is not the case.

Only (i) is feasible:

if $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{1}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon) = \varepsilon$ is a simple possibility.

Fix $\delta(\varepsilon) = \varepsilon$, then we get for BVP

$$Y''(\tau) + Y'(\tau) + \varepsilon Y'(\tau) + \varepsilon Y(\tau) = 0$$

(by multiplying BVP by $\delta = \varepsilon$):

So $\varepsilon \to 0$

$$Y''(\tau) + Y'(\tau) = 0$$

$$Y(0) = 0$$

has a solution

$$Y(\tau) = C_1 + C_2 e^{-\tau},$$

With $Y(0) = 0 \Rightarrow C_1 = -C_2$

$$\therefore\ Y(\tau) = C_1 (1 - e^{-\tau})$$

$$\therefore\ y_{inner}(t) = C_1 \left(1 - e^{-t/\varepsilon}\right)$$

Rmk: Recall that $y_{outer} = e^{1-t}$. We do have y_{inner} with a free parameter, C_1 and we'll adjust it so that y_{outer} & y_{inner} agree "in some sense".

We need an intermediate (small) region, between $O(\varepsilon)$ and $O(1)$. A reasonable guess is that

$$t = O(\sqrt{\varepsilon}) \therefore \text{ let } \eta = t/\varepsilon^{1/2}$$

We might propose that

$$\lim_{\varepsilon \to 0^+} y_{inner}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} y_{outer}(\sqrt{\varepsilon}\eta)$$

Rmk: we use this to determine C_1.

In our problem

$$\lim_{\varepsilon \to 0^+} y_{inner}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} C_1\left(1 - e^{-\eta/\sqrt{\varepsilon}}\right) = C_1$$

$$\lim_{\varepsilon \to 0^+} y_{outer}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} e^{1-\sqrt{\varepsilon}\eta} = e^1$$

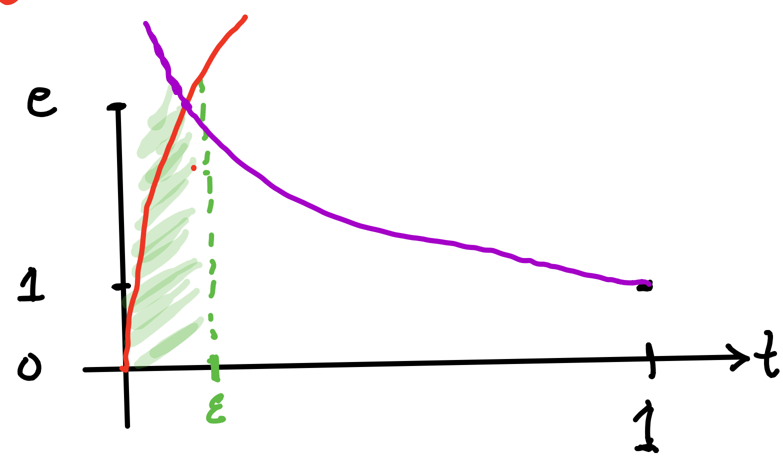
$$\therefore \quad C_1 = e^1$$

$$\therefore \quad y_{inner} = e\left(1 - e^{-t/\varepsilon}\right)$$

Rmk: One can construct a uniform approximation, valid for $0 \le t \le 1$ but this is beyond scope of this course. See Bender & Orszag, also Lin & Segel

In summary

$$\begin{cases} y_{outer} = e^{1-t} & t = O(1) \\ y_{inner} = e\left(1 - e^{-t/\varepsilon}\right) & t = O(\varepsilon) \end{cases}$$



Two Applications:

- SIMPLE HARMONIC OSCILLATOR
- CHEMICAL KINETICS

DAMPED (LINEAR) OSCILLATOR

$$\text{IVP} \begin{cases} m\ddot{y} + a\dot{y} + ky = 0 & t > 0 \\ y(0) = 0 \quad m\dot{y}(0) = I, \text{ constant} \end{cases}$$

$$(\dot{\ }) \equiv \frac{d}{dt}$$

Consider the case when <u>m is very small</u>

$t = [T]$ $\quad y = [L]$

$[m] = M \quad [a] = MT^{-1} \quad [k] = MT^{-2} \quad [I] = MLT^{-1}$

$$\begin{array}{c c} & \begin{array}{cccccc} t & y & m & a & k & I \end{array} \\ \begin{array}{c} T \\ L \\ M \end{array} & \begin{pmatrix} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Rank $r = 3$, 6 dimensional quantities $\therefore$

3 dimensionless groups

$$1 = [\pi] = [t^{\alpha_1} y^{\alpha_2} m^{\alpha_3} a^{\alpha_4} k^{\alpha_5} I^{\alpha_6}]$$

leads to

$$\alpha_1 - \alpha_4 - 2\alpha_5 - \alpha_6 = 0$$

$$\alpha_2 + \alpha_6 = 0$$

$$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \alpha_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_6 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Pi_1 = \frac{ta}{m} \qquad \Pi_2 = \frac{t^2 k}{m} \qquad \Pi_3 = \frac{tI}{ym}$$

So three possible time scales:

from Π_1 : $\frac{m}{a}$ from Π_2 : $\sqrt{\frac{m}{k}}$ from Π_3 : $\frac{a}{k}$

Π_3 uses length scales that appear below (see below for calc)

Substitute each of these for a scaling of time t

substitute Π_1 & Π_2 relations into Π_3.

These are 3 possible length scales:

$$\underline{\underline{\frac{I}{a}}}, \quad \frac{I}{\sqrt{km}}, \quad \underline{\underline{\frac{aI}{km}}}$$

Π_3 gave $\frac{a}{k}$ for the scales: $\Pi_3 = \frac{a}{k} B$, so $[B] = T^{-1}$

To find a B: we note that $\frac{\Pi_2^2}{\Pi_1} = \frac{m}{k}\frac{a}{m} = \frac{a}{k}$ $\therefore$

$$1 = [\Pi_3] = \left[\frac{\Pi_2^2}{\Pi_1} B\right] = \left[\frac{\Pi_2^2}{\Pi_1} \frac{I}{ym}\right] = T\left[\frac{I}{ym}\right] = T\frac{ML}{T}\frac{1}{L}\frac{1}{M} = T\frac{1}{T} = 1 \quad //$$

if $m \ll 1$ $\therefore$ Possible Time scale:

$\frac{m}{a} \ll 1$ and $\sqrt{\frac{m}{k}} \ll 1$, $\frac{a}{k}$ is balanced

Possible length scale:

$\frac{I}{\sqrt{mk}} \gg 1$, $\frac{aI}{km} \gg 1$, $\frac{I}{a}$ balanced.

We can achieve these. So, we suggest

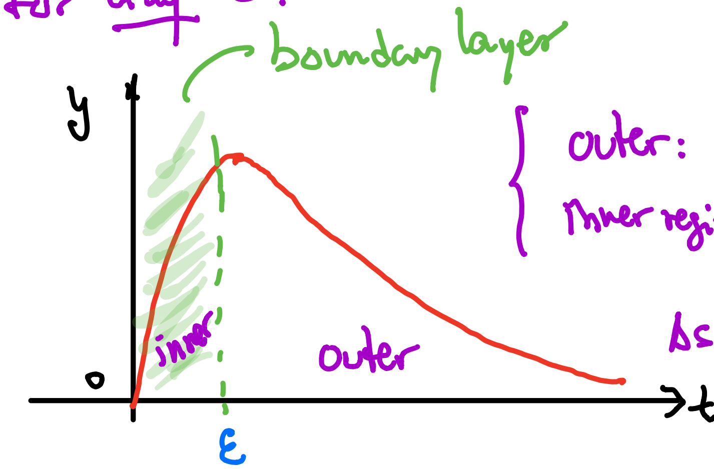
$$\therefore \quad \boxed{\tau = \frac{t}{a/k} \qquad Y = \frac{y}{I/a}}$$

With these, rescale the IVP:

$$\begin{cases} (\text{ODE}) \quad \varepsilon \dfrac{d^2Y}{d\tau^2} + \dfrac{d}{d\tau}Y + Y = 0 & \tau > 0 \\ \\ (\text{I.C.}) \quad Y(0) = 0 & \varepsilon Y'(0) = 1 \end{cases}$$

$$\varepsilon = \frac{mk}{a^2} \ll 1$$

<u>Rmk</u>: since this problem has an exact solution we already know what the dynamics are going to be for <u>any</u> ε.



outer: $m\ddot{y}$ small
inner region: $m\ddot{y}$ & $a\dot{y}$ dominant

As we saw from prior analyses.

Solving the IVP: assume that

$Y \propto e^{\beta\tau}$, replace in the (ODE) to find that

$$\varepsilon\beta^2 + \beta + 1 = 0$$ characteristic equation

$$\beta^2 + \frac{1}{\varepsilon}\beta + \frac{1}{\varepsilon} = 0$$

$$\beta_{1,2} = -\frac{1}{2\varepsilon} \pm \frac{1}{2}\sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}}$$

$$\beta_{1,2} = -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon} = -\frac{1}{2\varepsilon}\left(1 \mp \sqrt{1-4\varepsilon}\right)$$

if $4\varepsilon < 1$ then we get exponential growth & decay solutions (no oscillations)

The task is to find leading order approximation to the solution $y(t)$ for all time:

The leading order in the outer region when $\varepsilon = 0$

$$Y'_{outer} + Y_{outer} = 0 \qquad Y_{outer} = Ce^{-t}$$

Rmk: We will pin down C once we determine the inner solution & perform a stitching in an intermediate zone.

Inner solution: close to $t=0$. Propose

$$\hat{\tau} = \frac{\tau}{\delta(\varepsilon)} \qquad \hat{Y} = Y = y$$

the IVP:

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \frac{d^2\hat{Y}}{d\hat{\tau}^2} + \frac{1}{\delta(\varepsilon)} \frac{d}{d\hat{\tau}}\hat{Y} + \hat{Y}(\hat{\tau}) = 0$$

if $\delta(\varepsilon)=\varepsilon$, the first two terms will be $\gg 1$, the order of the last term

So, in the inner layer we propose:

$$\hat{\tau} = \frac{\tau}{\varepsilon} \qquad \hat{Y} = Y$$

$$\frac{d^2\hat{Y}}{d\hat{\tau}^2} + \frac{d}{d\hat{\tau}}\hat{Y} + \varepsilon\hat{Y} = 0$$

set $\varepsilon = 0 \Rightarrow \quad \frac{d^2\hat{Y}}{d\hat{\tau}^2} + \frac{d}{d\hat{\tau}}\hat{Y} = 0$

$$\hat{Y}(\hat{\tau}) = A + Be^{-\hat{\tau}}$$

$Y(\tau) = A + Be^{-\tau/\varepsilon}$ and

$$Y'(\tau) = -\frac{1}{\varepsilon}Be^{-\tau/\varepsilon}$$

Apply (I.C) $Y(\tau=0)=0$:

$$A=-B$$

$$\varepsilon Y'(0)=1 \Rightarrow \frac{1}{\varepsilon}=Y'(0)$$:

$$Y(\tau)=\left(1-e^{-\tau/\varepsilon}\right)$$

$$\therefore \quad Y_{inner}(\tau)=1-e^{-\tau/\varepsilon}$$

The overlap region:

$$\eta=\frac{\tau}{\sqrt{\varepsilon}}$$

The remaining constant is found by matching the inner & outer solutions in this intermediate region

$$\lim_{\varepsilon\to 0^+}\left(y_{outer}(\sqrt{\varepsilon}\eta)\right)=\lim_{\varepsilon\to 0^+}\left(y_{inner}(\sqrt{\varepsilon}\eta)\right)$$

or $\lim_{\substack{\varepsilon \to 0 \\ \eta \text{ fixed}}} C e^{-\sqrt{\varepsilon}\eta} = \lim_{\substack{\varepsilon \to 0 \\ \eta \text{ fixed}}} \left(1 - e^{-\eta/\sqrt{\varepsilon}}\right)$

$\therefore \quad C = 1$ //

Rmk: We can propose a uniformly valid approximation for the asymptotic solution as follows:

$$y_{uniform}(\tau) = y_{outer}(\tau) + y_{inner}(\tau) - \lim_{\substack{\varepsilon \to 0 \\ \eta \text{ fixed}}} y_{outer}(\tau)$$

$\therefore$ In terms of the (t, y) variables

$$y_{uniform}(t) = \frac{I}{a}\left(e^{-kt/a} - e^{-at/m}\right)$$ //

CHEMICAL KINETICS (Lin & Segel)

$$A + B \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_3} P + B$$

Reversible irreversible

For $t \geq 0$

$$\frac{da}{dt} = -k_1 ab + k_2 c \qquad a(0) = \hat{a} \qquad (1)$$

$$\frac{db}{dt} = -k_1 ab + k_2 c + k_3 c \qquad b(0) = \hat{b} \qquad (2)$$

$$\frac{dc}{dt} = k_1 ab - k_2 c - k_3 c \qquad c(0) = 0 \qquad (3)$$

$$\frac{dp}{dt} = k_3 c \qquad p(0) = 0 \qquad (4)$$

Assume $\hat{b} \ll \hat{a}$

* We note that (1) - (3) have no p

* We note that adding (2) & (3)

$$\frac{d}{dt}(b+c) = 0$$

$\therefore$ $b(t)+c(t) = $ constant for all t

at $b(0)+c(0) = \hat{b}$

$\therefore$ $\boxed{b(t)+c(t) = \hat{b}}$

Solving for $b(t) = \hat{b} - c(t)$ (✡)

We substitute (✡) into ① - ④

Ⓐ $\frac{da}{dt} = -k_1 a(\hat{b}-c) + k_2 c$ from ①

Ⓒ $\frac{dc}{dt} = k_1 a(\hat{b}-c) - (k_2+k_3)c$ from ③

For ④: $p = k_3 \int^t c(s)\,ds$

Rmk: for ① - ④: a quick decrease in a at first, then it levels off for longer times.

We expect c to decrease to zero in the long time limit. We infer the existence of a boundary layer near $t=0$.

We could set $\frac{da}{dt} = \frac{dc}{dt} = 0$

and find a stationary point $c=0$

and for small - : $c(0)=0$

So $\begin{cases} \frac{da}{dt} \sim -k_1 \hat{a} \hat{b} \\ \frac{dc}{dt} \sim k_1 \hat{a} \hat{b} \end{cases}$

let $\quad \bar{a} = \frac{a}{\hat{a}} \quad \bar{c} = \frac{c}{\hat{b}} \quad \bar{t} = \frac{t}{T}$

to be determined

Replace into Ⓐ, Ⓒ (and p eqs)

$$(*) \quad \frac{\hat{a}}{T} \frac{d\bar{a}}{d\bar{t}} = -k_1 \hat{a} \hat{b} \bar{a}(1-\bar{c}) + k_2 \hat{b} \bar{c}$$

$$(**) \quad \frac{\hat{b}}{T} \frac{d\bar{c}}{d\bar{t}} = k_1 \hat{a} \hat{b} \bar{a}(1-\bar{c}) - (k_2+k_3) \hat{b} \bar{c}$$

if $\dfrac{\hat{a}}{T}\dfrac{d\bar{a}}{d\bar{t}}$ balance with $-k_1\hat{a}\hat{b}\bar{a}(1-\bar{c})$ in (*)

then $T = \dfrac{1}{k_1\hat{b}}$ (‡)

We don't see any reason to find another dominant balance involving

$\dfrac{\hat{a}}{T}\dfrac{d\bar{c}}{d\bar{t}}$ and $k_2\hat{b}\bar{c}$, say.

Substitute for (‡) in (*) & (**)

(¥) $$\frac{d\bar{a}}{d\bar{t}} = -\bar{a} + (\bar{a}+\lambda)\bar{c} \qquad \lambda = \frac{k_2}{\hat{a}k_1}$$

$$\frac{\hat{b}}{\hat{a}}\frac{d\bar{c}}{d\bar{t}} = \bar{a} - (\bar{a}+\mu)\bar{c} \qquad \mu = \frac{k_2+k_3}{\hat{a}k_1}$$

let $\varepsilon \equiv \dfrac{\hat{b}}{\hat{a}} \ll 1$

$\bar{a}(0) = 1 \qquad \bar{c}(0) = 0$

$$(\$) \qquad \varepsilon \frac{d\bar{c}}{d\bar{t}} = \bar{a} - (\bar{a} + \mu)\,\bar{c}$$

Set $\varepsilon = 0$

$$\frac{d\bar{a}_{outer}}{d\bar{t}} = -\bar{a}_{outer} + (\bar{a}_{outer} + \lambda)\,\bar{c}_{outer}$$

$$0 = \bar{a}_{outer} - (\bar{a}_{outer} + \mu)\,\bar{c}_{outer}$$

$\bar{c}_{outer}$ found from ($), setting $\varepsilon \frac{d\bar{c}}{d\bar{t}} = 0$:

$$\boxed{\bar{c}_{outer} = \frac{\bar{a}_{outer}}{\bar{a}_{outer} + \mu}}$$

$$\frac{d\bar{a}_{outer}}{d\bar{t}} = -\bar{a}_{outer} + (\bar{a}_{outer} + \lambda)\,\frac{\bar{a}_{outer}}{(\bar{a}_{outer} + \mu)}$$

$$\left(1 + \frac{\mu}{\bar{a}_{outer}}\right)\frac{d\bar{a}_{outer}}{d\bar{t}} = -(\mu - \lambda)$$

Integrating this separable ODE:

$$(\yen)\quad \bar{a}_{outer} + \mu \ln \bar{a}_{outer} = (\lambda-\mu)\bar{t} + K$$

Rmk: $\lambda - \mu < 0$

Rmk: $(\yen)$ is an implicit eq for $\bar{a}_{outer}$.

K will be set in the matching procedure

To find $\bar{a}_{inner}$, valid for $t \sim 0$, let

$$\tau = \frac{\bar{t}}{\delta(\varepsilon)} \qquad A = \bar{a} \qquad C = \bar{c}$$

then substitute into $(\pounds)$ & $(\$)$:

$$\frac{1}{\delta(\varepsilon)}\frac{dA}{d\tau} = -A + (A+\lambda)C$$

$$\frac{\varepsilon}{\delta(\varepsilon)}\frac{dC}{d\tau} = A - (A+\mu)C$$

set $\delta(\varepsilon) = \varepsilon$ $\therefore$

$$\frac{1}{\varepsilon}\frac{dA}{d\tau} = -A + (A+\lambda)C$$

$$\Rightarrow \frac{dA}{d\tau} = -\varepsilon A + \varepsilon(A+\lambda)C$$

$$(\#)\quad \frac{dC}{d\tau} = A - (A+\mu)C.$$

Set $\varepsilon = 0$ we get $\frac{dA}{d\tau} = 0$ or $A = \text{constant}.$

Since $A(0) = 1 \Rightarrow A = 1$

Now solve for C inside layer. Use $A = 1$ in $(\#)$

$$\frac{dC}{d\tau} = 1 - (1+\mu)C \quad \text{A linear ODE.}$$

$$\begin{cases} \bar{C}_{inner} = \frac{1}{1+\mu} + M e^{-(\mu+1)\bar{t}/\varepsilon} \\ \bar{a}_{inner} = 1 \end{cases}$$

We have 2 unknowns (K, M): use I.C. on C

to pin down μ:

$\bar{c}_{inner}(0) = 0 \quad \therefore$

$$\begin{cases} \bar{c}_{inner} = \frac{1}{\mu+1}\left(1 - e^{-(\mu+1)\bar{t}/\varepsilon}\right) \\ \bar{a}_{inner} = 1 \end{cases}$$

Now, we pin down K in a region intermediate between 0 and 1:

Let $\eta = \bar{t}/\sqrt{\varepsilon}$

$$\lim_{\varepsilon \to 0^+} \bar{a}_{outer}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} \bar{a}_{inner}(\sqrt{\varepsilon}\eta) = 1 \quad (*)$$

$$\lim_{\varepsilon \to 0^+} \bar{c}_{outer}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon \to 0^+} \bar{c}_{inner}(\sqrt{\varepsilon}\eta) \quad (**)$$

Evaluate $(*)$

$$1 + \mu \ln(1) = K \quad \therefore K = 1$$

Evaluate (**)

$$(***) \quad \lim_{\varepsilon\to 0^+} \bar{C}_{inner}(\sqrt{\varepsilon}\eta) = \frac{1}{\mu+1}$$

$$(*4) \quad \lim_{\varepsilon\to 0^+} \bar{C}_{outer}(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0^+} \frac{\bar{a}_{outer}(\sqrt{\varepsilon}\eta)}{\bar{a}_{outer}(\sqrt{\varepsilon}\eta)+\mu}$$

Since

$$\bar{a}_{outer} + \mu \ln \bar{a}_{outer} = (\lambda-\mu)\bar{t} + K$$

Match (***) w/ (*4)

$$\lim_{\varepsilon\to 0^+} \frac{\bar{a}_{outer}(\sqrt{\varepsilon}\eta)}{\bar{a}_{outer}(\sqrt{\varepsilon}\eta)+\mu} = \frac{1}{\mu+1}$$

Rmk: Since $\bar{a}_{outer}$ is an implicit equation, you will need to proceed further using numerical means.

To obtain a uniformly valid approximation

$$\bar{a}_{uniform} = \bar{a}_{outer}(\bar{t}) + \bar{a}_{inner}(\bar{t}) - 1 = \bar{a}_{outer}(\bar{t})$$

$$\bar{c}_{uniform} = \bar{c}_{outer}(\bar{t}) + \bar{c}_{inner} - \frac{1}{1+\mu}$$

To find $\bar{a}_{outer}$ you need to solve the implicit formula

$$\bar{a}_{outer}(\bar{t}) + \mu \ln\left(\bar{a}_{outer}(\bar{t})\right) = -(\mu-\lambda)\bar{t} + 1$$

so that

$$\lim_{\bar{t}\to\infty} \bar{a}_{outer}(\bar{t}) = \lim_{\bar{t}\to\infty} e^{-\frac{(\mu-\lambda)\bar{t}}{\mu}}$$

$$\therefore \lim_{\bar{t}\to\infty} \bar{a}_{outer}(\bar{t}) = 0$$

With a, c known, b & p follow.

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