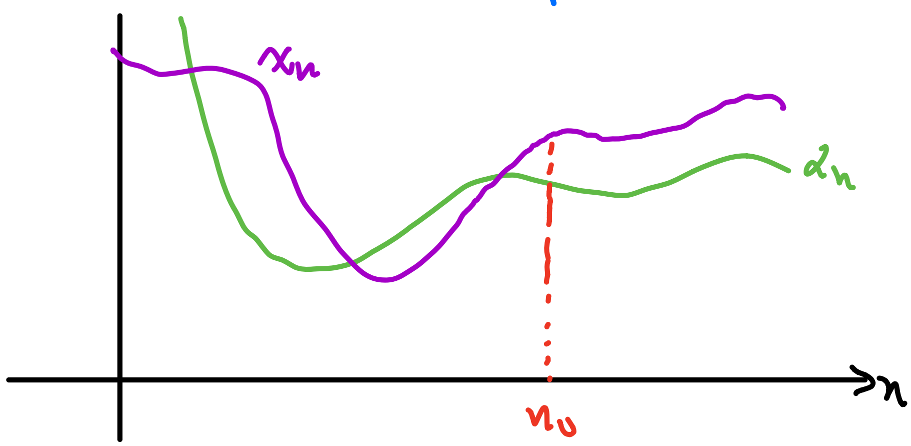
Big O & little o Notation

Given 2 sequences
$[x_n]$ and $[\alpha_n]$, $n = 0, 1 \dots$

We write $x_n = O(\alpha_n)$ if there exists
a constant K and $n = n_0$ s.t.

$$|x_n| \leq K|\alpha_n| \quad \text{for } n \geq n_0$$



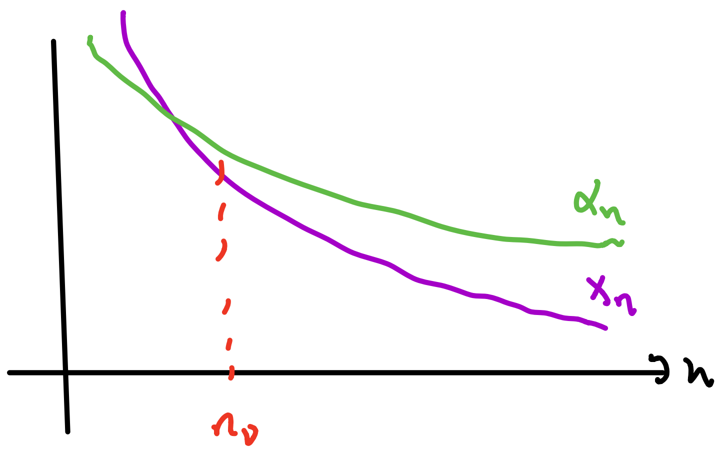
$$\lim_{n \to \infty} \left| \frac{x_n}{\alpha_n} \right| \leq K$$

We write $x_n = o(\alpha_n)$ if for each $\varepsilon > 0$
there exists an n_0 such that

$$0 \le |x_n| \le |\alpha_n| \varepsilon$$

when $n \ge n_0$

i.e. $\lim_{n \to \infty} \left| \frac{x_n}{\alpha_n} \right| = 0$



Rmk: could also let $\varepsilon = 1/n$ and then have statements regarding sequence behaviour as $\varepsilon \to 0$.

ex) Verify that $\sin x = O(x)$ as $x \to 0^+$

The statement "$f(\varepsilon) = O(1)$" means that f is bounded in a neighborhood of $\varepsilon \sim 0$.

$$\lim_{\varepsilon\to 0} \frac{\sin\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\cos\varepsilon}{1} = 1$$

by L'Hopital's Rule

Could have also used the mean value theorem on the derivative

$$\frac{\sin\varepsilon - \sin 0}{\varepsilon - 0} = \cos a \le 1, \text{ for some } a.$$

$$\therefore |\sin\varepsilon| = |\varepsilon\cos a| \le |\varepsilon| \quad \text{since } |\cos a| \le 1.$$

Also could have use Taylor series: recall that

$$\sin\varepsilon = \varepsilon - \frac{1}{3!}\varepsilon^3 + \frac{1}{5!}\varepsilon^5 - \dots .$$

divide both sides by ε and take

ex) Verify that $\varepsilon^2 \ln\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$

The statement "$f(\varepsilon) = o(1)$" means that

$f(\varepsilon) \to 0$ as $\varepsilon \to 0$

L'Hopital's rule.

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon^2 \ln\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\ln(\varepsilon)}{1/\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{1/\varepsilon}{(-1/\varepsilon^2)} = 0 \quad //$$

PERTURBATION METHODS

There are many methods. Here we pick two of the basic ones: REGULAR & SINGULAR Methods.

Rmk: Most problems do not have an analytic solution.
You are left with

- Computational approximations
- local perturbative approaches (Approximation)

PERTURBATION APPROACHES:

Some challenges with these approaches are:

1. Often not possible to quantify to what extent the approximation reflects the behavior of the true solution.
However, in some cases, the perturbative

approach, which is highly local, can capture the dynamics of the system globally.

2. It might lead to generalizations about the true system that are significantly different from true dynamics.

Rmk: in summary, exercise caution when applying the methodology.

Regular Perturbation Methods

For specificity, consider the

$$\text{IVP}\begin{cases} \dfrac{dy}{dt} = f(y,t;\varepsilon), & t>0 \\ y(t=0) = y_0 \end{cases}$$

$$y \in \mathbb{R}^n, \quad f \in \mathbb{R}^n, \quad y_0 \in \mathbb{R}^n$$

$\varepsilon \ll 1$ is a parameter that is either explicit in the IVP, or is introduced.

Rmk: We will explore what happens to $y(t)$ for $\varepsilon \ll 1$, but could also use $\delta = \frac{1}{\varepsilon}$ and look at $y(t)$ when $\delta \to \infty$.

Rmk: We are going to assume in what follows that $y = O(1)$ $\frac{dy}{dt} = O(1)$

$y_A = O(1)$

WLOG consider the scalar case:

$$\begin{cases} \frac{dy}{dt} = f(y,t), \quad t>0 & (\ast) \\ y(0) = y_A \end{cases}$$

Assume that

$$y(t) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \quad (\dagger)$$

Call $y_0(t)$ the leading order of $y(t)$

Replace (‡) into (★)

Rmk: ε is not a precise number, merely a constant $<<1$

$$\frac{d}{dt}\left(y_0+\varepsilon y_1+\varepsilon^2 y_2\cdots\right)=f\left(y_0+\varepsilon y_1+\varepsilon^2 y_2\cdots,t\right) \quad (\exists)$$

Rmk: the idea is that $y_1, y_2 \ldots$ are "corrections" to the leading order term y_0. Note $y_i = O(1)$

Assume that $\frac{\partial^n f}{\partial y^n}$ exist, so the right hand side of $(\exists)$ can be written as a series about y_0. So the right hand side is

$$f\left(y_0+\varepsilon y_1+\varepsilon^2 y_2\cdots,t\right)=f(y_0,t)+\varepsilon\left.\frac{\partial f}{\partial y}\right|_{y=y_0} y_1+O(\varepsilon^2)$$

The left hand side, on the other hand, is

$$\frac{d}{dt}\left(y_0 + \varepsilon y_1 + \dots\right) = \frac{dy_0}{dt} + \varepsilon \frac{dy_1}{dt} + \varepsilon^2 \frac{dy_2}{dt} \dots$$

Match both sides of the equation, by orders in ε:

ε^0: $$\frac{dy_0}{dt} = f(y_0, t)$$

ε^1: $$\frac{dy_1}{dt} = \left.\frac{\partial f}{\partial y}\right|_{y=y_0} y_1$$

ε^2: $$\frac{dy_2}{dt} = \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{y=y_0} y_2$$

etc...

Next, solve the ε^0 equation to obtain y_0 (you have to be able to solve this equation. If not, the method fails.

Next, solve the ε^1 equation to obtain y_1 (and so on...)

$$y(t) \approx y_0 + \varepsilon y_1 \dots \text{, truncated}$$

is the approximate solution to (✱)

It remains valid, so long as $\varepsilon \ll 1$

ex) Motion of an object in a nonlinear resistive medium.

A body of mass m, is given an initial velocity V_0, and moves in a medium that offers a resistive force of magnitude $av - bv^2$. Here, $v = v(t)$ is the velocity of body, time is denoted by t and a and b are positive constants.

$$\begin{cases} m\dfrac{dv}{dt} = -av + bv^2 \\ v(0) = V_0 \end{cases}$$

The velocity scale is V_0 and for dimensionless time $\tau = \dfrac{t}{m/a}$

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = \frac{m}{a}\frac{d}{dt}$$

let $y = \dfrac{V}{V_0}$

$$(\star)\quad \begin{cases} \dfrac{dy}{d\tau} = -y + \varepsilon y^2 & \tau > 0 \\ y(0) = 1 \end{cases}$$

where $\varepsilon = \dfrac{bV_0}{a} \ll 1$

by assumption.

$(\star)$ is a Bernoulli Equation

Rmk: The Bernoulli Equations

$$\frac{dy}{dx} + p(x)y = q(x)\,y^n$$

Trick to solving is to let $w = y^{1-n}$

and follow the consequences...

here $n = 2$

(A) $w = y^{-1}$

$$\frac{dw}{d\tau} = \frac{d}{d\tau}\left(y^{-1}(\tau)\right) = -y^{-2}\frac{dy}{d\tau} \quad (B)$$

substitute (A) & (B) into $(\cancel{A})$

$$-\frac{1}{y^{-2}}\frac{dw}{d\tau} = -w^{-1} + \varepsilon w^{-2} \quad \therefore$$

$$\frac{dw}{d\tau} = -y^{-2}\left(-w^{-1} + \varepsilon w^{-2}\right) = w - \varepsilon$$

$$\therefore \quad \frac{dw}{d\tau} = w - \varepsilon$$

<u>Rmk:</u> this is a linear 1st order ODE

$$\frac{dw}{d\tau} - w = -\varepsilon$$

use an integrating factor $I = e^{-\tau}$

$$e^{-\tau}\frac{dw}{d\tau} - e^{-\tau}w = -\varepsilon e^{-\tau}$$

$$\frac{d}{d\tau}\left(e^{-\tau}w\right) = -\varepsilon e^{-\tau}$$

$$we^{-\tau} = -\varepsilon \int_0^{\tau} e^{-s}\, ds + c$$

$$w = -\varepsilon e^{\tau} \int_0^{\tau} e^{-s} ds + e^{\tau} c$$

recall that $y(0) = 1 \Rightarrow w(0) = 1$

$\therefore \quad \frac{1}{y} = w = -e^{\tau} + 1 + ce^{\tau}$ or

$$y_{exact} = \frac{e^{-\tau}}{1 + \varepsilon(e^{-\tau} - 1)}$$

We're going to use perturbation method and see how the perturbation solution compares to y_{exact}:

Rmk: Look at (☆) set $\varepsilon = 0$:

$$(\star)\ \begin{cases} \dfrac{dy}{d\tau} = -y + \cancel{\varepsilon}y^2 = -y \\ y(0) = 1 \end{cases}$$

That has the solution $\tilde{y}(\tau) = e^{-\tau}$ ($)

of $\dfrac{d\tilde{y}}{d\tau} = -\tilde{y} \quad \tilde{y}(0) = 1$

should be in some sense an approximation to

$y_{exact}(\tau)$

Compare ($) to y_{exact} when $\varepsilon = 0$ and we confirm this. Also, take y_{exact} and use a Taylor series approximation ($\varepsilon \ll 1$):

$$y_{exact} = e^{-\tau} + \varepsilon(e^{-\tau} - e^{-2\tau}) + \varepsilon^2(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau}) + \dots$$

we see that to leading order $y_{exact} = \tilde{y}$

$$\therefore \quad y_{exact} = \tilde{y} + G(\varepsilon)$$

Let's obtain the approximate solution via regular perturbation series:

Let $y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) \cdots$ (¥)

then substitute (¥) into (A), i.e. into:

$$\begin{cases} \frac{dy}{d\tau} = -y + \varepsilon y^2 \\ y(0) = 1 \end{cases}$$

and then organize by powers of ε:

$$\frac{d}{d\tau}\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots\right) = -\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots\right) + \varepsilon\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)^2$$

$$\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)(0) = 1\varepsilon^0 + 0\varepsilon^1 + 0\varepsilon^2 + \cdots$$

Order in powers of ε:

$$\varepsilon^0: \begin{cases} \frac{d}{d\tau} y_0 = -y_0 \\ y_0(0) = 1 \end{cases} \Rightarrow y_0 = e^{-\tau}$$

$$\varepsilon^1: \begin{cases} \dfrac{d}{d\tau} y_1 = -y_1 + y_0^2 = -y_1 + e^{-2\tau} \\ y_1(0) = 0 \end{cases}$$

$$\text{or } \begin{cases} \dfrac{dy_1}{d\tau} + y_1 = e^{-2\tau} \\ y_1(0) = 0 \end{cases} \Rightarrow \quad y_1 = e^{-\tau} - e^{-2\tau}$$

$$\varepsilon^2: \begin{cases} \dfrac{dy_2}{d\tau} = -y_2 + 2y_0 y_1 = -y_2 + 2e^{-\tau}\left(e^{-\tau} - e^{-2\tau}\right) \\ y_2(0) = 0 \end{cases}$$

or

$$\begin{cases} \dfrac{dy_2}{d\tau} + y_2 = 2\left(e^{-2\tau} - e^{-3\tau}\right) \\ y_2(0) = 0 \end{cases} \qquad y_2(\tau) = e^{-\tau} - 2e^{-\tau} + e^{-3\tau}$$

y_p is the perturbation solution, to $O(\varepsilon^3)$:

$$y_p(\tau) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \overset{?}{\approx} y_{exact} \text{ to } O(\varepsilon^3)$$

$$y_p(\tau) = e^{-\tau} + \varepsilon\left(e^{-\tau} - e^{-2\tau}\right) + \varepsilon^2\left(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau}\right)$$

$y_p(\tau)$, to leading order is the same as y_{exact} for $\varepsilon = 0$.

Compare $y_{exact} - y_p$

$$y_{exact} = e^{-\tau} + \varepsilon(e^{-\tau} - e^{-2\tau}) + \varepsilon^2(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau}) + \ldots$$

$$y_p = e^{-\tau} + \varepsilon(e^{-\tau} - e^{-2\tau}) + \varepsilon^2(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau})$$

$$\therefore \quad y_{exact} - y_p = O(\varepsilon^3)$$

Rmk: It would seem that perturbation series is a generally-applicable method, leading to satisfactory outcomes every time. The more common situation, however, is that perturbation solutions are at best satisfactory under very restrictive conditions. In some cases the solution may be plain wrong!

What do we mean by "satisfactory"?

The ideal notion of success would be

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \left| y_{exact}(t) - \sum_{i=0}^{N} \varepsilon^i y_i \right| = 0$$

for arbitrary t. So satisfactory would be, for example that $y_0 \approx y_{exact}$ for some range of ε, with errors $O(\varepsilon)$, $\varepsilon \ll 1$.

This is called asymptotic convergence.

Realistically, you at least expect qualitative agreement and that the more terms you add to the approximation the higher the qualitative fidelity.

NONLINEAR OSCILLATOR

We will see that regular perturbation methods will yield the wrong result. We'll then introduce a different perturbation method that gives acceptable results, the Poincaré Linstedt Method

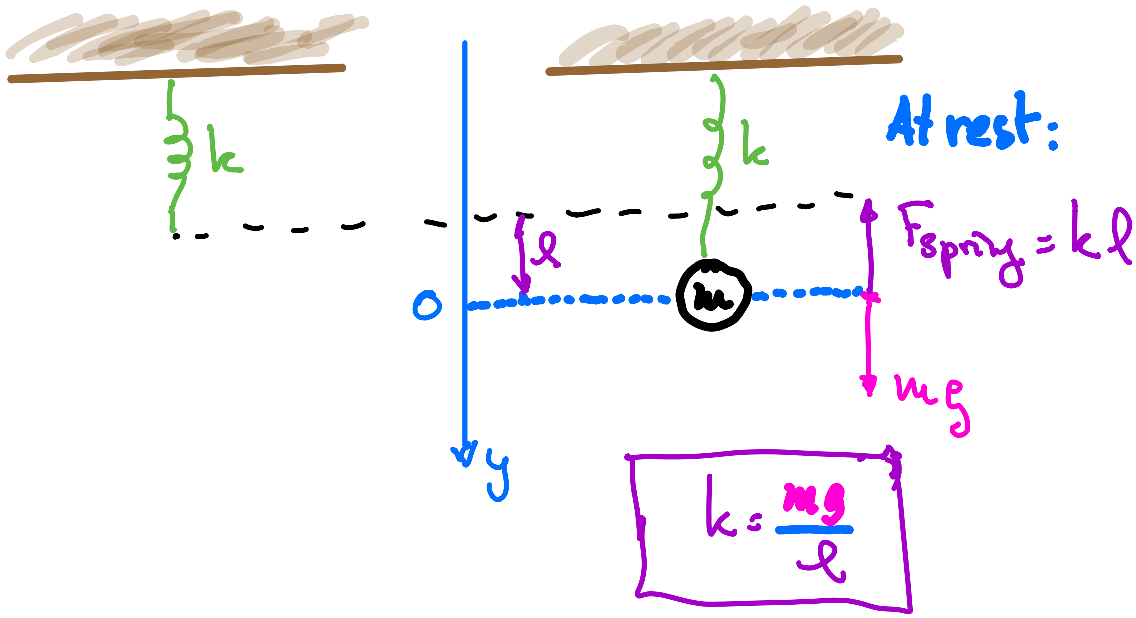
The linear oscillator assumes Hooke's Law is valid:

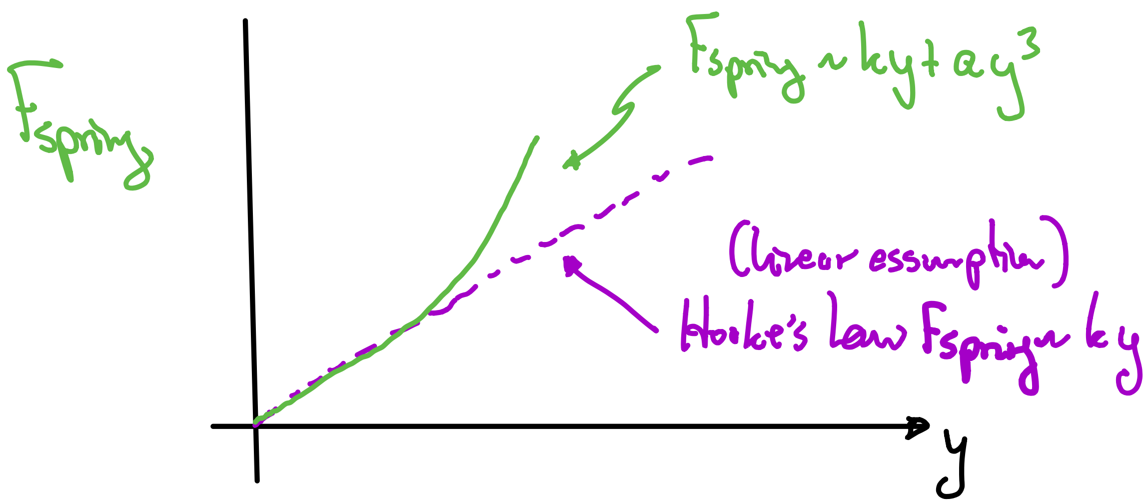
$$m \frac{d^2y}{d\tau^2} = -ky$$

τ is time

y is the amplitude or displacement

k is the "spring constant", m is the mass of the oscillator





Hooke's Law: assume spring force is linear with displacement, assumed to be approximately correct when $|y| << 1$, small.

For large y: typical springs will have nonlinear displacement dependence in creating a force.

A common model:

$F_{spring} = ky + ay^3$

$a << k$

Consider

ODE $\quad m\dfrac{d^2y}{d\tau^2} = -ky - ay^3 \qquad \tau > 0$

I.C. $\quad y(0) = A \qquad \dfrac{dy}{d\tau}(0) = 0$

NONDIMENSIONALIZATION:

$[A] = L \qquad [m] = M \qquad [k] = \dfrac{M}{T^2}$

$[\tau] = T \qquad [a] = \dfrac{M}{L^2T^2}$

$\omega_0^2 = \dfrac{k}{m}$ is the natural freq

$[\omega_0] = \dfrac{1}{T}$

$$u = \frac{y}{A} \qquad t = \frac{\tau}{\sqrt{m/k}} = \omega_0 \tau$$

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = \omega_0 \frac{d}{dt}$$

$$\frac{d^2}{d\tau^2} = \omega_0^2 \frac{d^2}{dt^2}$$

$$\begin{cases} \dfrac{mA}{\omega_0^2}\dfrac{d^2}{dt^2}u = -kAu - aA^3u^3 \\[2ex] Au(0) = A \qquad \dfrac{A}{\omega_0}\dfrac{d}{dt}u(0) = 0 \end{cases}$$

some algebra:

$$(\dagger) \quad \begin{cases} \dfrac{d^2u}{dt^2} + u + \varepsilon u^3 = 0 & \text{(ODE)} \\[2ex] u(0) = 1 \qquad u'(0) = 0 & \text{(I.C.)} \end{cases}$$

$$\varepsilon = \frac{aA^2}{k} \quad (\ll 1 \text{ by assumption})$$

The usual regular perturbation method:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) \cdots$$

Then $(\ddagger)$:

(ODE) $$\frac{d^2}{dt^2}\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \cdots\right) + \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\right. $$
$$\left. + \varepsilon\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\right)^3 = 0\right.$$

(IC) $(u_0 + \varepsilon u_1 \cdots)(0) = 1 \qquad \frac{d}{dt}(u_0 + \varepsilon u_1 + \cdots) = 0$

Collect terms in terms of powers of ε:

ε^0: $$\begin{cases} \dfrac{d^2 u_0}{dt^2} + u_0 = 0 \\ u_0(0) = 1 \qquad u_0'(0) = 0 \end{cases}$$

ε^1: $$\begin{cases} \dfrac{d^2 u_1}{dt^2} + u_1 = -u_0^3 \\ u_1(0) = u_1'(0) = 0 \end{cases}$$

ε^2: $$\begin{cases} \dfrac{d^2 u_2}{dt^2} + u_2 = 3u_0^2 u_1 \\ u_2(0) = u_2'(0) = 0 \end{cases}$$

etc.

The ε^0 solution $u_0(t) = \cos t$

$$u_1'' + u_1 = -\cos^3 t = -\frac{1}{4}(3\cos t + \cos 3t)$$

$$u_1 = u_1^h + u_1^p = c_1\cos t + c_2 \sin t + u_1^p$$

Use the MUC (method of undetermined coefficients) to find u_1^p:

$$u_1^p = C\cos 3t + D\,t\cos t + E\,t\sin t$$

because $u_1^h = (\cos t, \sin t)$

Substitute u_1^p into the ε^1 equation, you get

$$u_1 = c_1 \cos t + c_2 \sin t + \frac{1}{32}\cos 3t - \frac{3}{8}t\sin t$$

Apply I.C. to find c_1 & c_2

$$\therefore u_1 = \frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8}t\sin t$$

$\therefore$ To $O(\varepsilon^2)$

$$u_{approx} = u_0 + \varepsilon u_1 + O(\varepsilon^2)$$

$$= \underbrace{\cos t}_{u_0} + \varepsilon \left[\underbrace{\frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8} t \sin t}_{u_1} \right]$$

$$+ O(\varepsilon^3)$$

To leading order we get $u_0 = \cos t$, as expected, a good solution. The u_1 correction, however grows unbounded by as $t \to \infty$, even if $\varepsilon \ll 1$

Is this sensible?

Let's examine $(\dagger)$

$$\ddot{u} + u + \varepsilon u^3 = 0$$

and calculate the energy: multiply by $\frac{1}{2}\dot{u}$

$$\frac{1}{2}\dot{u}\left(\ddot{u} + u + \varepsilon u^3\right) = 0$$

or

$$\frac{d}{dt}\left(\dot{u}^2 + u^2 + \frac{\varepsilon}{2}u^4\right) = 0$$

Integrate both sides

$$\dot{u}^2 + u^2 + \frac{\varepsilon}{2}u^4 = \text{Constant}$$

use $u(0) = 1$ $\quad \dot{u}(0) = 0$ at $t = 0$

then at $t = 0$ $\quad 1 + \frac{1}{2}\varepsilon = \text{Constant}$

$\therefore \quad \dot{u}^2 + u^2 + \frac{1}{2}\varepsilon u^4 = 1 + \frac{1}{2}\varepsilon > C$, constant

for all t

$$\dot{u} = \pm\sqrt{1 + \tfrac{1}{2}\varepsilon - u^2 - \tfrac{1}{2}\varepsilon u^4}$$

where $\quad 0 \leq u^2 + \frac{1}{2}\varepsilon u^4 \leq 1 + \frac{1}{2}\varepsilon$

So the solution of $\ddot{u} + u + \varepsilon u^3 = 0$ must be bounded (i.e. the energy at $t = 0$ is the same for all t, and the potential energy $u^2 + \frac{1}{2}\varepsilon u^4$ is bounded, and so $\dot{u}$

Rmk: The conclusion is that something is wrong with the regular perturbation approximation to the solution. Maybe the problem is that we did not take enough terms in our perturbation expansion? (you can see what happens, by computing the $\varepsilon^2, \varepsilon^3$, etc terms). But we can also guess that in order for the 1st correction, the term proportional to ε to remain bounded, that we'd need to require that $\varepsilon \sim 1/t$.

The only recourse is to say that the regular perturbation solution is OK for a limited time span $t \in [0, T]$ such that

$$\varepsilon\left[\frac{1}{2}(\cos 3t - \cos t) - \frac{3\varepsilon}{8} t \sin t\right] \text{ remains}$$

small, which requires that

$$\frac{3\varepsilon t}{8} < 1 \Rightarrow T \sim 8/3\varepsilon \quad //$$

Rmk: The u_{1p} is called a "secular" term and these will arise in the application of regular perturbation series applied to all oscillatory problems.

We wound up with a secular term because of "resonances":

In our series

$$\ddot{u}_0 + u_0 = 0 \rightarrow u_0 = \cos t$$

$$\ddot{u}_1 + u_1 = -u_0^3 \rightarrow \text{lead to } u_1 = u_H + u_p$$

where $u_p \propto t$.

$$\ddot{u}_2 + u_2 = \text{——}$$

$$\vdots$$

let $\mathcal{L} \equiv \dfrac{d^2}{dt^2} + 1$

then at all orders $i = 0, 1, \ldots$

$$\mathcal{L} u_i = f(u_{i-1} \ldots)$$

So the u_{i-1}, u_{i-2}, etc will have terms in the null space of the operator $\mathcal{L}$:

$\mathcal{L} u_1 = 0$ solutions

$\therefore$ these will blow up $\propto t$.

The Poincaré-Linstedt Method

Is a modification of the regular perturbation method that is suitable for oscillatory problems, like the nonlinear oscillator.

The idea is to perturb BOTH the amplitude as well as the frequency.

Let $u(\tau) = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) \ldots$ (A)

where $\tau = \omega t$

Let $\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ldots$ (B)

this was equal to 1 in the $\ddot{u} + u + \varepsilon u^3 = 0$

problem.

substitute (A) & (B) into IVP (‡)

i.e. $\begin{cases} \ddot{u} + u + \varepsilon u^3 = 0 \\ u(0) = 1 \quad u'(0) = 0 \end{cases}$

yields: for $u = u(\tau)$

$$\begin{cases} \omega^2 \dfrac{d^2u}{d\tau^2} + 1u + \varepsilon u^3 = 0 & \tau > 0 \\ u(0) = 1 & \dfrac{du}{d\tau}(0) = 0 \end{cases}$$

this would be ω_0^2, which happens to be 1 in this problem

$$\frac{du(t)}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt}u = \omega\frac{du}{d\tau}, \text{etc.}$$

$$\left(\omega_0^2 + 2\varepsilon\omega_1\omega_0 + \cdots\right)\left(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' \cdots\right)$$

(this would be ω_0^2)

$$1\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \cdots\right)$$

$$+\, \varepsilon\left(u_0^3 + 3\varepsilon u_0^2 u_1 + \cdots\right) = 0$$

also $\begin{cases} u_0(0) + \varepsilon u_1(0) + \cdots = 1 \\ u_0'(0) + \varepsilon u_1'(0) \cdots = 0 \end{cases}$

Order in ε:

ε^0: $\quad u_0'' + \underset{1''}{\omega_0} u_0 = 0 \quad u_0(0) = 1 \quad u_0'(0) = 1$

ε^1: $\quad u_1'' + \underset{1''}{\omega_0} u_1 = -2\omega_1 u_0'' - u_0^3 \quad u_1(0) = u_1'(0) = 0$

etc

The solution to ε^0: $\quad u_0 = \cos\tau$

To ε^1: $\quad u_1'' + \overset{1''}{\omega_0} u_1 = 2\omega_1 \cos\tau - \cos^3\tau$

$$= \left(2\omega_1 - \frac{3}{4}\right)\cos\tau - \frac{1}{4}\cos 3\tau$$

we set $\omega_1 = \frac{3}{8}$ in order to eliminate the appearance of the secular term in the u_1 solution.

$$u_1'' + u_1 = -\frac{1}{4}\cos 3\tau$$

with general solution

$$u_1(\tau) = c_1\cos\tau + c_2\sin\tau + \frac{1}{32}\cos 3\tau$$

Apply I.C. $u_1(0) = u_1'(0) = 0$

$$\therefore u_1(\tau) = \frac{1}{32}(\cos 3\tau - \cos\tau)$$

$$\therefore u(\tau) = \cos\tau + \frac{1}{32}\varepsilon(\cos 3\tau - \cos\tau) + O(\varepsilon^2)$$

and $\tau = t + \frac{3}{8}\varepsilon t + \ldots\ldots$ //

Poincare-Linstedt applied to Vander Pol Oscillator:

$$\ddot{x} + \mu\omega_0(x^2-1)\dot{x} + \omega_0^2 x = 0$$

$\frac{d}{dt} = \dot{(\ )}$ $\quad \mu \ll 1$

$x'(0) = 0 \quad x(0) = A$

let $\tau = \omega t$ then

$$\omega^2 x'' + \mu\omega_0\omega(x^2-1)x' + \omega_0^2 x = 0 \qquad (\ddagger)$$

$(\ )' = \frac{d}{d\tau}$

where $\omega = \omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots\cdots$

$x(\tau) = x_0 + \mu x_1 + \mu^2 x_2 + \cdots\cdots$

substituting into (‡)

$$\left(\omega_0^2 + 2\mu\omega_1\omega_0 + \mu^2(\omega_1^2 + 2\omega_0\omega_2) + \cdots\right)\left(x_0'' + \mu x_1'' + \cdots\right)$$

$$+ \mu\omega_0\left(\omega_0 + \mu\omega_1 + \mu^2\omega_2 + \cdots\right)\left[\left(x_0 + \mu x_1 + \mu^2 x_2 \cdots\right)^2 - 1\right]\left(x_0' + \mu x_1' + \cdots\right)$$

$$+ \omega_0^2\left(x_0 + \mu x_1 + \mu^2 x_2 + \cdots\right) = 0$$

And I.C.:

$$\left(x_0 + \mu x_1 + \mu^2 x_2 \cdots\right)(0) = A$$

$$\left(x_0 + \mu x_1 + \mu^2 x_2 \cdots\right)'(0) = 0$$

Separate in orders in μ:

μ^0: $\omega_0^2\left(x_0'' + x_0\right) = 0$

$$x_0(0) = A \quad x_0'(0) = 0$$

$$\Rightarrow x(\tau) = \alpha\cos(\tau + \beta)$$

applying I.C.

$$x_0(\tau) = A\cos\tau$$

μ^1: $x_1'' + x_1 = -2\frac{\omega_1}{\omega_0}x_0'' + (1 - x_0^2)x_0'$

$$x_1(0) = x_1'(0) = 0$$

so substituting x_0:

$$x_1'' + x_1 = 2A\frac{\omega_1}{\omega_0}\cos\tau - (1 - A^2\cos^2\tau)A\sin\tau$$

$$= 2A\frac{\omega_1}{\omega_0}\cos\tau - A\left(1 - A^2 + \frac{3}{4}A^2\right)\sin\tau$$
$$+ \frac{1}{4}A^3\sin 3\tau$$

$\therefore$ $A\omega_1 = 0 \Rightarrow \omega_1 = 0$

$$A\left(1 - \frac{1}{4}A^2\right) = 0 \Rightarrow A = 2$$

$\therefore$ $x_1'' + x_1 = \frac{1}{4}A^3\sin 3\tau$ with $A = 2$

$$x_1 = A_1\cos\tau + B_1\sin\tau - \frac{1}{4}\sin 3\tau$$

since $x'(0) = 0 \Rightarrow B_1 = \frac{3}{4}$

then A_1 is left undetermined

To find A_1, need to go to next order.

Rmk: It is tempting to use $x_1(0) = 0$ to sort out A_1. You will find that this leads to an inconsistency at $O(\mu^2)$.

μ^2: $$x_2'' + x_2 = \frac{5}{4\omega_0^2}\cos 5\tau - \frac{3}{2\omega_0^2}\cos 3\tau + 3A_1 \sin 3\tau + \left(4\frac{\omega_2}{\omega_0} + \frac{1}{4\omega_0^2}\right)\cos\tau + 2A_1\sin\tau$$

So $\omega_2 = -\frac{1}{16\omega_0}$ and $A_1 = 0$

$$\therefore \quad x(\tau) = 2\cos\tau + \frac{3}{4}\mu\sin\tau - \frac{1}{4}\mu\sin 3\tau + O(\mu^2)$$

$$\omega = \omega_0\left(1 - \frac{1}{16}\frac{\mu^2}{\omega_0^2}\right) + O(\mu^3)$$

Rmk: Another useful perturbation method for oscillatory problems is called the method of MULTIPLE SCALES

Rmk: We employed the series

$$y_0(t) + \varepsilon y_1(t) + \dots$$

i.e. powers in the small parameter ε.

But in some problems the series might be

$$y_0(t) + \varepsilon^{1/2} y_1(t) + \varepsilon y_2(t) \dots$$

or

$$y_0(t) \ln\varepsilon + y_1(t)\varepsilon\ln\varepsilon + y_2(t)\varepsilon^2\ln\varepsilon + \dots$$

It depends on the problem. In general these are all s.t.

$$g_{n+1}(t,\varepsilon) = o\left(g_n(t,\varepsilon)\right) \text{ as } \varepsilon \to 0$$

i.e. each successive term is strictly smaller than the one before //.