ORTHOGONALITY OF FUNCTIONS

Two functions $f(x)$ & $g(x)$, complex or real, are said to be "orthogonal with respect to the weight $w(x)$ over an interval $I=[a,b]$" if

$$\int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx = 0$$

where $\overline{(\ )}$ is complex conjugate.

The weight $w(x) \geq 0$ for $x \in (a,b)$ and

$$\int_a^b w(x)\,dx = A \geq 0, \text{ a number.}$$

ex) Under what conditions are $f_m(x) = e^{imx}$ and $f_n(x) = e^{inx}$ orthogonal with respect to $w(x)=1$ over $I = [-\pi, \pi]$?

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx}\, dx = \int_{-\pi}^{\pi} e^{i(m-n)x}\, dx$$

$$= \frac{1}{i(m-n)} e^{i(m-n)x}\Big|_{-\pi}^{\pi} = \frac{1}{i(m-n)}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right]$$

$$= \frac{2}{(m-n)} \sin\left[(m-n)\pi\right]$$

Case (i) $m \neq n$ let $p = m-n$

$f(p) \equiv \frac{2}{p} \sin p\pi$ if p is not an integer, then $f(p) \neq 0$.

if $p = \pm 1, \pm 2, \pm 3 \cdots$, $f(p) = 0$.

if $p = 0$? $\lim_{p \to 0} \frac{2}{p} \sin p\pi = \lim_{p \to 0} 2\pi \cos p\pi = 2\pi$ by L'Hopital's rule.

$\therefore$ $\varphi_n(x)$ & $\varphi_m(x)$ are orthogonal if $p \equiv m-n$ is $p = \pm 1, \pm 2, \pm 3 \ldots$

Hence, if $m = \alpha n + \beta$, where α is a natural number and $|\beta| < n$

then we showed that $\beta = 0$ leads to p being an integer $(\neq 0)$.

So $p = m - n = \alpha n - n = (\alpha - 1)n$

when φ_n, φ_m orthogonal and $\alpha \neq 1$

$\therefore$ φ_n, φ_m are orthogonal UNLESS $n = m$, where $m, n = \pm 1, \pm 2, \pm 3 \ldots$

When $m = n$

$$\int_{-\pi}^{\pi} \varphi_n \bar{\varphi}_m 1 \, dx = 2\pi$$

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Sturm-Liouville Problem

A special boundary value problem: linear and of even order. A special property is that it generates ORTHOGONAL FUNCTIONS

We focus on 2nd order case: consider $y = y(x)$

such that $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda w(x)\right] y = 0 \qquad (\ddagger)$$

on a bounded interval $x = a$, $x = b$ plus 2 boundary conditions.

p, q, w are real.

let $\mathcal{L} \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$, then $(\ddagger)$ is

$$(*) \qquad \mathcal{L}y + \lambda w y = 0 \quad .$$

An aside: Consider the general second order linear eq

$$(**) \qquad a_0(x)y'' + a_1(x)y' + \left[a_2(x) + \lambda a_3(x)\right]y = 0$$

if $p = e^{\int a_1/a_0\, dx}$ $\quad q = \frac{a_2}{a_0}p \quad w = \frac{a_3}{a_0}$ then $(**)$ can be written as $(*)$. //

We will show that solutions to SL problems, with different λ values, have the property that

$$\int_a^b w(x)\,\varphi_i\,\varphi_j\,dx = 0$$

where $\begin{cases} \varphi_i \text{ is the solution to } \mathcal{L}\varphi_i + \lambda_i\varphi_i = 0 \\ \varphi_j \text{ " " " } \mathcal{L}\varphi_j + \lambda_j\varphi_j = 0 \end{cases}$

To see this:

① $$\frac{d}{dx}(p\,\partial_x\varphi_i) + (q + \lambda_i w)\varphi_i = 0$$

② $$\frac{d}{dx}(p\,\partial_x\varphi_j) + (q + \lambda_j w)\varphi_j = 0$$

multiply ① by φ_j and ② by φ_i and subtract:

$$\varphi_j(p\,\partial_x\varphi_i)_x - \varphi_i(p\,\partial_x\varphi_j)_x + (\lambda_i - \lambda_j)\,w\,\varphi_i\varphi_j = 0$$

or

$$(\lambda_i - \lambda_j)\,w\,\varphi_i\varphi_j = \varphi_i(p\,\partial_x\varphi_j)_x - \varphi_j(p\,\partial_x\varphi_i)_x$$

integrate by parts:

$$(\lambda_i - \lambda_j)\int_a^b w(x)\,\varphi_i\varphi_j\,dx = p\partial_x\varphi_j\varphi_i\Big|_a^b - \int_a^b p\,\varphi_{ix}\varphi_{jx}\,dx$$
$$- p\partial_x\varphi_i\varphi_j\Big|_a^b + \int_a^b p\,\varphi_{ix}\varphi_{jx}\,dx$$

or
$$(\lambda_i - \lambda_j)\int_a^b w(x)\,\varphi_i\varphi_j\,dx = \left(p\left[\partial_x\varphi_j\varphi_i - \partial_x\varphi_i\varphi_j\right]\right|_a^b$$

So $\int_a^b w(x)\,\varphi_i\varphi_j\,dx = 0$ depending on boundary conditions (RHS) on φ_i & φ_j at $x=a$, $x=b$:

CASE (A) if $\varphi = 0$ at $x=a,b$ $\quad \int_a^b w\,\varphi_i\varphi_j\,dx = 0$

if $\varphi_x = 0$ at $x=a,b$ $\quad \int_a^b w\,\varphi_i\varphi_j\,dx = 0$

if $\varphi + \gamma\varphi' = 0$ at $x=a,b$: $\quad \int_a^b w\,\varphi_i\varphi_j\,dx = 0$

CASE (B) if $p = 0$ at $x=a$, $x=b$ $\quad \int_a^b w\,\varphi_i\varphi_j\,dx = 0$

if φ is finite at $x=a,b$ and φ or $p\varphi'$ tend to 0 at $x=a$, $x=b$.

CASE (C) if $p(b) = p(a)$ $\quad \int_a^b w\,\varphi_i\varphi_j\,dx = 0$ if

~~$\varphi_i(a)$~~ $\varphi(a) = \varphi(b)$ and $\varphi'(a) = \varphi'(b)$, which is satisfied when φ is periodic, with period $b-a$. //

ex) $(\star)\begin{cases} y'' + \lambda^2 y = 0 & \text{or} \quad \frac{d}{dx}[y_x] + \lambda^2 y = 0 \\ y(0) = y(l) = 0 & \text{the weight } w = 1 \end{cases}$

Solutions of $\star$ $\quad y = C\varphi_n$, where $\varphi_n = \sin\frac{n\pi x}{l}$

and $\lambda_n = \frac{n\pi}{l} \quad n = 1, 2, \dots$

$(\star)$ is a Sturm-Liouville problem with orthogonal solutions. That is,

$$\int_0^l w(x)\varphi_n(x)\varphi_m(x)\,dx = \int_0^l 1 \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$

$$= \begin{cases} l/2 & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}$$

In many instances we find it convenient to normalize the functions φ_n: let

$$\hat{\varphi}_n(x) = \sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l} \quad \text{then}$$

$$\int_0^l 1\,\hat{\varphi}_n(x)\hat{\varphi}_m(x)\,dx = \delta_{nm}$$

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$\hat{\varphi}_n(x)$ are "orthonormal".

Expansion of an Arbitrary function in a series of Orthogonal functions, on a fixed interval $I=[a,b]$.

Suppose we know a family of functions $\varphi_n(x)$ which are orthogonal on $I=[a,b]$ with respect to $w(x)$, a weight $w(x)\geq 0$ & $\int_a^b w(x)dx = A \geq 0$.

We want to express

$$f(x) = \sum_{n=0}^{\infty} A_n \varphi_n(x)$$

as a linear combination of these functions, A_n are constants. If $\varphi_n(x)$ are orthogonal

$$w(x) f(x) = \sum_{n=0}^{\infty} A_n w(x) \varphi_n(x)$$

$$w(x) \varphi_m(x) f(x) = \sum_{n=0}^{\infty} A_n w(x) \varphi_n(x) \varphi_m(x)$$

Integrate both sides from a, b:

$$\int_a^b w(x) \varphi_m(x) f(x) dx = \sum_{n=0}^{\infty} A_n \int_a^b w(x) \varphi_n(x) \varphi_m(x) dx$$

$$= A_m Q \quad \text{when } m=n$$

$$= 0 \quad \text{when } m\neq n$$

So

$$A_m = \frac{1}{Q_m}\int_a^b w(x)\,\varphi_m(x)\,f(x)\,dx \qquad m = 0, 1, 2, \ldots$$

$$Q = \int_a^b w(x)\,\varphi_m^2(x)\,dx$$

SOME FAMILIES OF ORTHOGONAL FUNCTIONS (ALL OF THESE HAVE ASSOCIATED STURM-LIOUVILLE PROBLEMS!)

family	interval	weight $w(x)$
$\sin\frac{n\pi x}{\ell}, \cos\frac{n\pi x}{\ell}$	$x = [-\frac{\ell}{2}, \frac{\ell}{2}]$	1
Legendre	$-1 \le x \le 1$	1
Chebyshev	$-1 \le x \le 1$	$(1-x^2)^{-1/2}$
Hermite	$-\infty < x < \infty$	e^{-x^2}
Bessel $J_0(x)$	$0 \le x < \infty$	x

etc

(SL) Sturm-Liouville, Revisited:

$$\mathcal{L} y = -\lambda w y \quad \text{, plus boundary conditions}$$

$$-\frac{1}{w}\mathcal{L} y = \lambda y \quad \text{or} \quad Ly = \lambda y$$

$$\text{where } \mathcal{L} = \left[p(x)\frac{d}{dx}\right]_x + q(x)$$

Looks a lot like the matrix eigenvalue problem

$$A\mathbf{z} = \lambda \mathbf{z}$$

A is an $n \times n$ matrix z is a vector λ is a constant. We call z the eigenvectors and λ the eigenvalues.

In fact the solutions y of the SL problem are called eigenfunctions, λ are the eigenvalues. L is the infinite dimensional analogue of A

if $w(x) > 0$ on $[a,b]$
$p(x) > 0$ on $[a,b]$

and the boundary conditions chosen so that

$$\int_a^b w\,\varphi_n \varphi_m \,dx = 0 \qquad n \neq m$$

$$\int_a^b w\,\varphi_n^2 \,dx > 0 \qquad n = m$$

then we say that $L \equiv \frac{1}{w}\left[p\frac{d}{dx}\right]_x + \frac{q}{w}$

is "self-adjoint". Which means that

* All eigenfunctions are real

(**) Eigenfunctions belonging to different eigenvalues λ are orthogonal with respect to $w(x)$ on $[a,b]$

(***) It has an infinity many eigenvalues (and corresponding eigenfunctions).

(****) The eigenvalues can be arranged in an increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 \cdots \lambda_n <$$

$\lambda_n \to \infty$ as $n \to \infty$.

ex) let $f(x) = x$

expand $f(x)$ in a series, derived from the SL problem

$$SL \begin{cases} y'' + k^2 y = 0 \\ y(0) = y(l) = 0 \end{cases}$$

in $x \in [0, l]$:

SL has solutions $\varphi_n = \sin\frac{n\pi x}{l}$ $\quad k_n = \frac{n\pi}{l}$ $\quad n = 1, 2, \ldots$

these are orthogonal wrt $w(x) = 1$ on $[0, l]$

so $$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

$$w\,\varphi_m(x) f(x) = \sum_{n=0}^{\infty} a_n \varphi_m \varphi_n w \qquad w = 1$$

integrate ~~wrt x~~

$$\int_0^l \varphi_m(x) f(x)\,dx = \sum_{n=0}^{\infty} a_n \int_0^l \varphi_m \varphi_n\,dx = a_m \int_0^l \sin^2\frac{m\pi x}{l}\,dx$$

$$\int_0^l \varphi_m(x)\, x\,dx = a_m \frac{l}{2}$$

$$\therefore\ a_m = \frac{2}{l}\int_0^l x \sin\frac{m\pi x}{l}\,dx$$

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$$a_m = -\frac{2l}{m\pi}\cos m\pi = \frac{2l}{m\pi}\frac{(-1)^{m+1}}{m} \quad m = 1, 2 \ldots$$

$$\therefore \; x = \sum_{n=1}^{\infty} a_n \varphi_n = \sum_{m=1}^{\infty} \frac{2l}{\pi}\frac{(-1)^{m+1}}{m} \sin\frac{m\pi x}{l}$$