NON HOMOGENEOUS BVP

Given the following problem to solve

$$\text{BVP}\quad \begin{cases} -[p(x)y']' + qy = \mu r(x) y + f(x) \\ \qquad\qquad a<x<b \\ \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

Rmk: This is a NONHOMOGENEOUS ODE

$$\mathcal{L}y \equiv -[p(x)y']' + q(x)y$$

Solve BVP

$$(£)\quad y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

b_n are unknown

$$(*)\quad \begin{cases} \mathcal{L}\phi_n = r(x)\lambda_n \phi_n \\ \text{B.C.} \end{cases}$$

The idea is to use (✡) to generate the necessary $\phi_n(x)$. These happen to be eigenfunctions of a SL w/ eigenvalues λ_n

1. Find the $\phi_n(x)$ & λ_n of the ✡
2. Compute the coefficients c_n's of

$$(\#) \qquad f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) r(x)$$

Multiply b.s. of (#) by $\phi_m(x)$ & integrate between $x=a$, $x=b$:

$$\int_a^b \phi_m(x) f(x)\, dx = \int_a^b dx\, \phi_m(x) \sum_{n=1}^{\infty} c_n \phi_n(x) r(x)$$

$$= \sum_{n=1}^{\infty} c_n \int_a^b dx\, r(x) \phi_m(x) \phi_n(x)$$

because $\displaystyle\int_a^b r(x) \phi_m \phi_n \, dx = \begin{cases} N^2 & n=m \\ 0 & n \neq m \end{cases}$

$$\therefore \quad \int_a^b f(x)\,\phi_m(x)\,dx = c_m N^2$$

$$\Rightarrow \quad c_m = \frac{1}{N^2}\int_a^b f(x)\,\phi_m(x)\,dx$$

3 Substitute ($\not{*}$) into BVP and also (#)

$$\underline{\mathcal{L}}\sum_{n=1}^{\infty} b_n \underline{\phi_n(x)} = \mu r(x)\sum_{n=1}^{\infty} b_n\phi_n + \sum_{n=1}^{\infty} c_n\phi_n(x)\,r(x)$$

$$\sum_{n=1}^{\infty} b_n \underbrace{\mathcal{L}\phi_n(x)}_{\text{by } (\not{*})} = \sum_{n=1}^{\infty}\mu b_n r(x)\,\phi_n + \sum_{n=1}^{\infty} c_n\phi_n(x)\,r(x)$$

$$\sum_{n=1}^{\infty} b_n\, r(x)\,\lambda_n\phi_n = \sum_{n=1}^{\infty}\mu b_n r(x)\,\phi_n + \sum_{n=1}^{\infty} c_n\phi_n(x)\,r(x)$$

Multiply B.S. by $\phi_m(x)$ & integrate

$$\sum_{n=1}^{\infty} b_n\lambda_n\int_a^b r(x)\,\phi_m\phi_n\,dx$$

$$= \sum_{n=1}^{\infty} \mu b_n \int_a^b r(x) \phi_m \phi_n \, dx + \sum_{n=1}^{\infty} c_n \int_a^b \phi_n \phi_n r(x) \, dx$$

$$\therefore$$

$$b_m \lambda_m = \mu b_m + c_m$$

so solve for $b_m = \dfrac{c_m}{\lambda_m - \mu}$

$m = 1, 2 \ldots$