QUALITATIVE ANALYSIS OF DYNAMICS

(Ch 9 Boyce & DiPrima)

We want qualitative insights into the solutions of a system of equations of the form

$$\begin{cases} \dfrac{d\underset{\sim}{y}}{dt} = \underset{\sim}{f}(\underset{\sim}{y}, t) & t > 0 \\ \underset{\sim}{y}(0) = \underset{\sim}{y}_0 \end{cases}$$

here $\underset{\sim}{y}, \underset{\sim}{y}_0$ and $\underset{\sim}{f}$ are $\mathbb{R}^n$

Perhaps we'd like to determine whether a solution is stable or not.

Background: Take $(\dagger)$ $\dfrac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} \quad t > 0$

$$\underset{\sim}{x} \in \mathbb{R}^2 \quad A \in \mathbb{R}^{2\times 2}$$

A is a constant matrix.

The solution of (‡) is found by computing the 2 eigenvectors and associated eigenvalues

$$\therefore \quad \underset{\sim}{x}(t) = c_1 e^{\lambda_1 t} \underset{\sim}{v}_1 + c_2 e^{\lambda_2 t} \underset{\sim}{v}_2$$

where $\lambda_{1,2}$ are the roots of the quadratic equation (characteristic):

$$p(\lambda) = \det(A - \lambda_i I) = 0 \quad i=1,2$$

the e'vectors are non-trivial solution

$$(A - \lambda_i I) v_i = 0 \quad i=1,2$$

There are 3 basic outcomes, for the root finding problem, each leading to a different temporal behavior.

We examine these next:

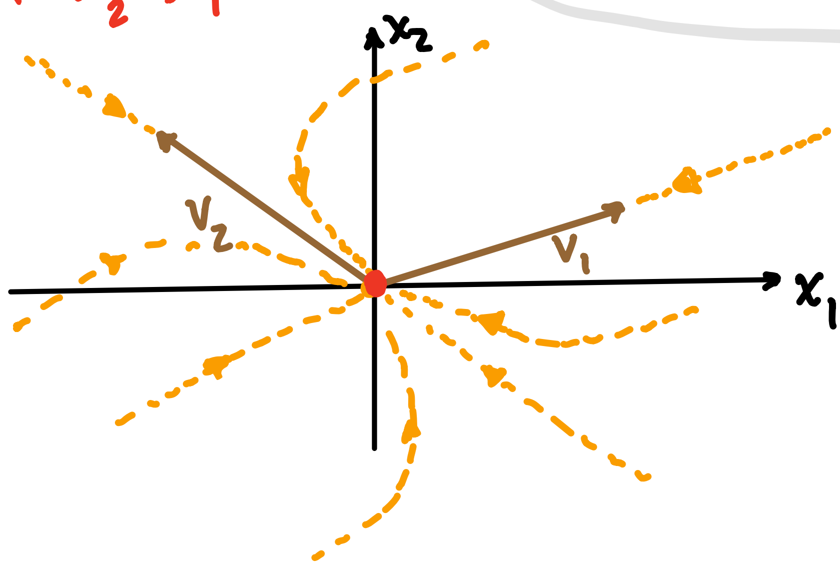
Assume $t>0$ in what follows:

REAL UNEQUAL ROOTS (E'VALUES)

(A) **Same sign case:** both either positive or negative real roots.

$$\underset{\sim}{x} = c_1 e^{\lambda_1 t} \underset{\sim}{v_1} + c_2 e^{\lambda_2 t} \underset{\sim}{v_2}$$

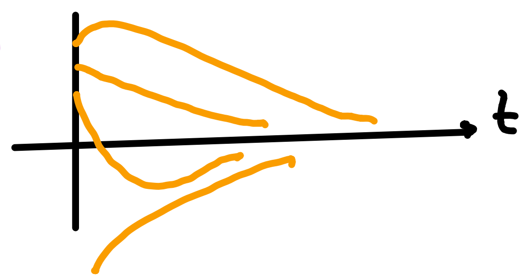
if $\lambda_2 < \lambda_1 < 0$:



$\lim_{t \to \infty} \underset{\sim}{x}(t) = \underset{\sim}{0}$ We say that $(0,0)$ is a "STABLE NODAL POINT"

<u>Rmk:</u> No matter what initial conditions you choose $\underset{\sim}{x}(0) = (x_1(0), x_2(0))$, if you wait long enough, the solution will asymptote to $(0,0)$.

Rmk: $x_i(t)$



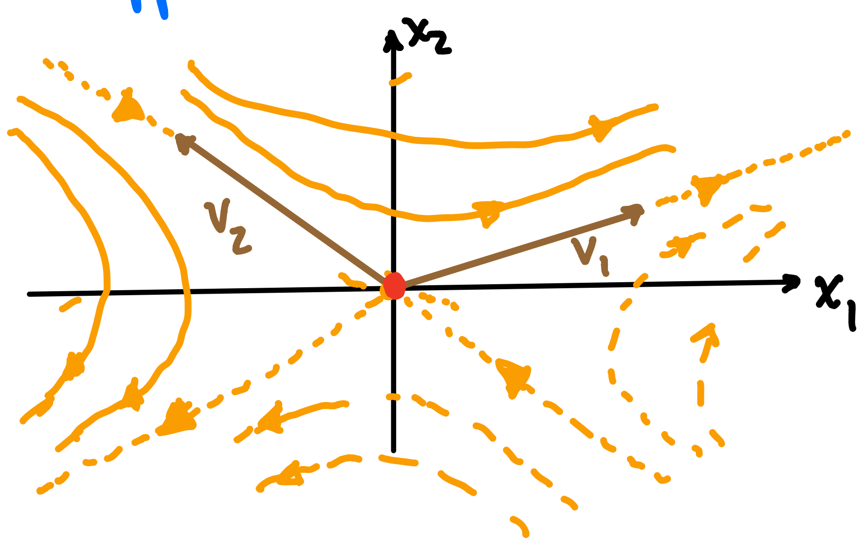
if $\lambda_1 > \lambda_2 > 0$: then $\lim_{t\to\infty} \underset{\sim}{x}(t) = \underset{\sim}{\infty}$

Similar picture to the one above but with trajectories effectively leaving from $\underset{\sim}{x} \approx \underset{\sim}{0}$ (unless $\underset{\sim}{x} = 0$)

UNSTABLE NODAL POINT

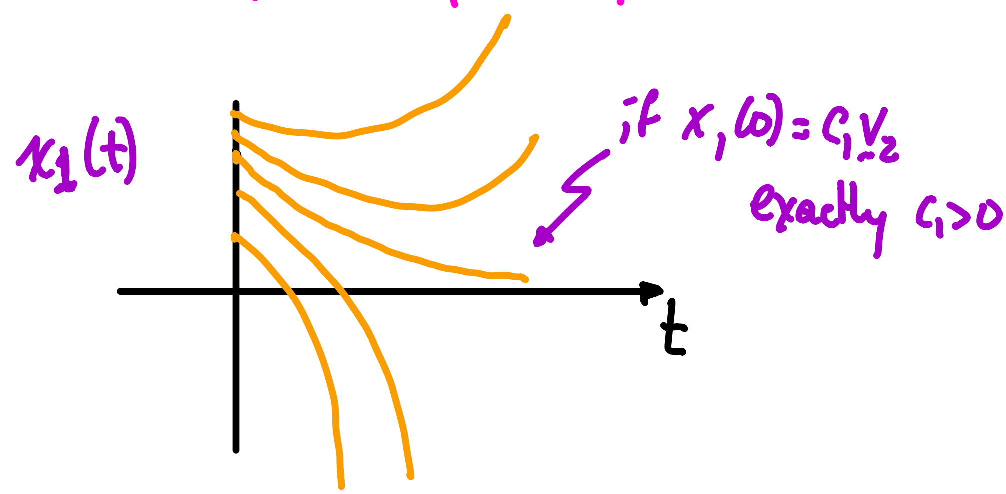
(B) Real and Unequal with Opposite Sign:

Suppose $\lambda_1 > 0$, $\lambda_2 < 0$



SADDLE POINT

<u>Rule:</u>



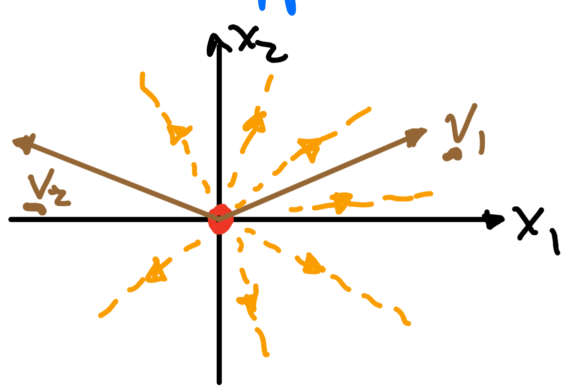
EQUAL ROOTS (E'VALUES)

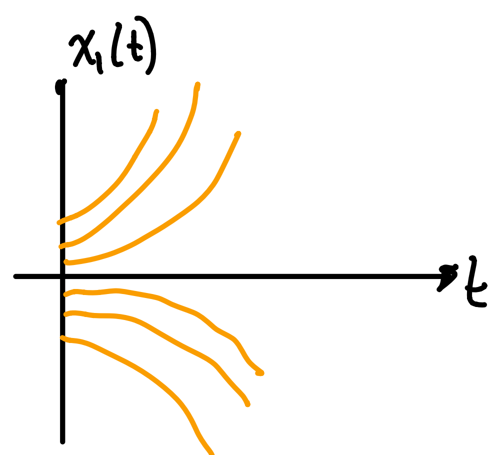
$$\lambda_1 = \lambda_2 \equiv \lambda$$

(A) Two independent eigenvectors

$$\underset{\sim}{x}(t) = e^{\lambda t}\left[c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2\right]$$

Suppose $\lambda > 0$





UNSTABLE PROPER NODE

Suppose $\lambda < 0$

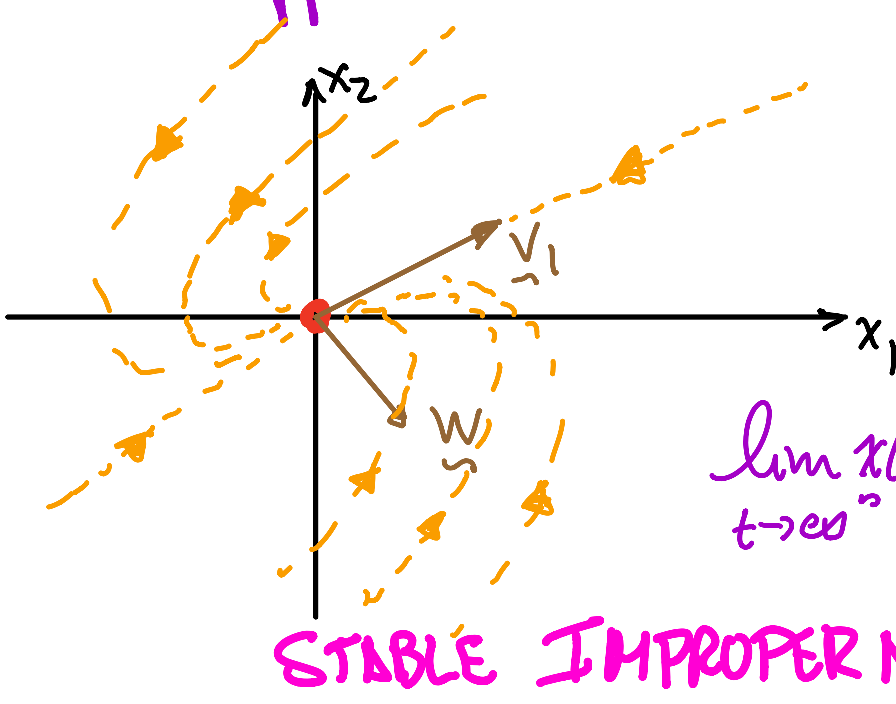
As above but now every solution will tend to 0 as $t \to \infty$.

STABLE PROPER NODE

(B) Only v_1 is found $\lambda_1 = \lambda_2 = \lambda$

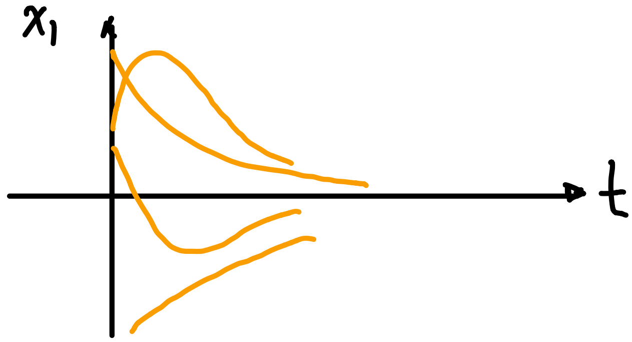
$$x(t) = c_1 \underset{\sim}{v_1} e^{\lambda t} + c_2 (\underset{\sim}{v} t + \underset{\sim}{w}) e^{\lambda t}$$

Suppose $\lambda < 0$



$$\lim_{t \to \infty} \underset{\sim}{x}(t) = \underset{\sim}{0}$$

STABLE IMPROPER NODE



SUPPOSE $\lambda > 0$ arrows turn away from $(0,0)$.

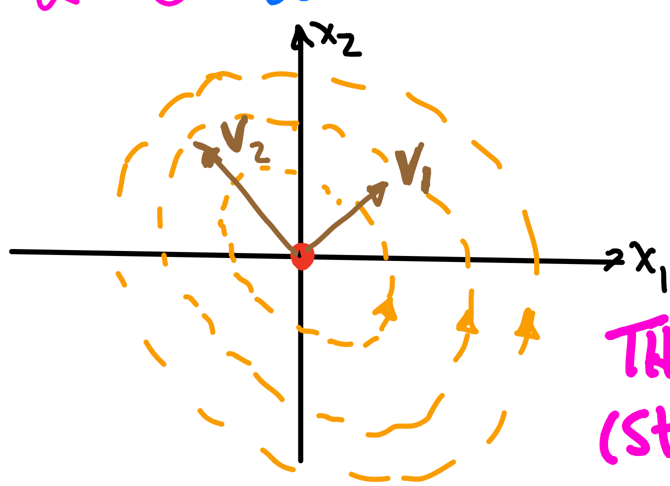
UNSTABLE IMPROPER NODE.

COMPLEX ROOTS (E'VALUES)

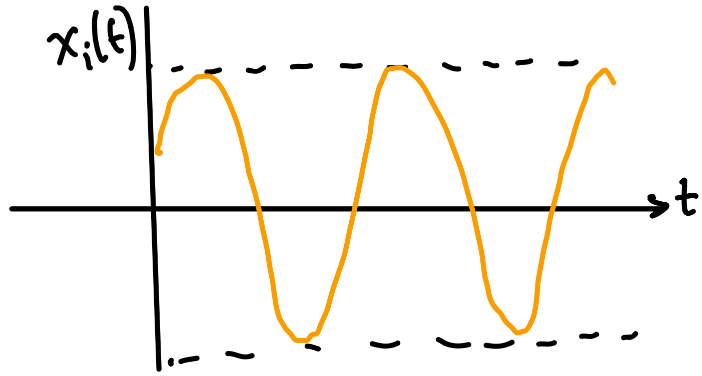
these will be $\lambda_{1,2} = \alpha \pm i\beta$

complex conjugates

(A) $\alpha = 0$ Solutions are sinusoidal



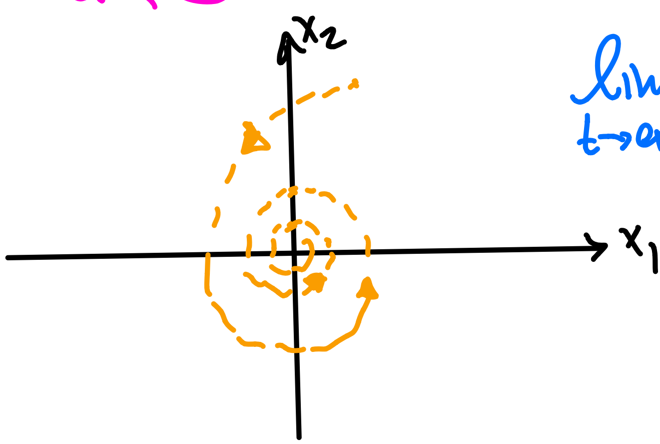
THIS IS A "CENTER"
(Stable)

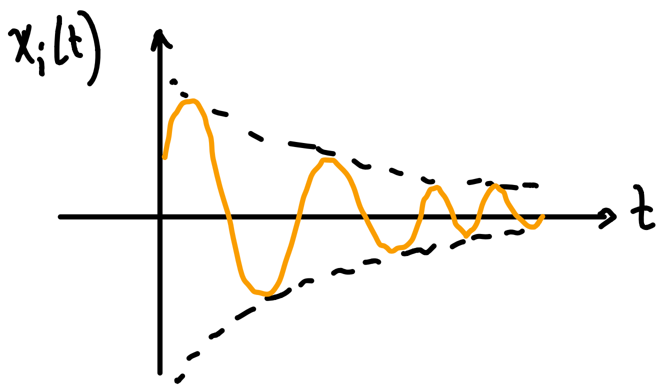


$x_{1,2}(t)$ are sinusoidal in time

(B) $\alpha < 0$ **STABLE SPIRAL**

$$\lim_{t\to\infty} x(t) = 0$$





(C) $\alpha > 0$ **UNSTABLE SPIRAL**

$\lim_{t\to\infty} x(t)$ diverges.

It looks like the above spiral, but arrows are reverse. The solution oscillates as it grows exponentially,

Def: A system of ODEs

$$\frac{dy}{dt} = \underset{\sim}{f}(y,t)$$

is called **autonomous** if

$$\underset{\sim}{f}(y,t) = \underset{\sim}{f}(y)$$

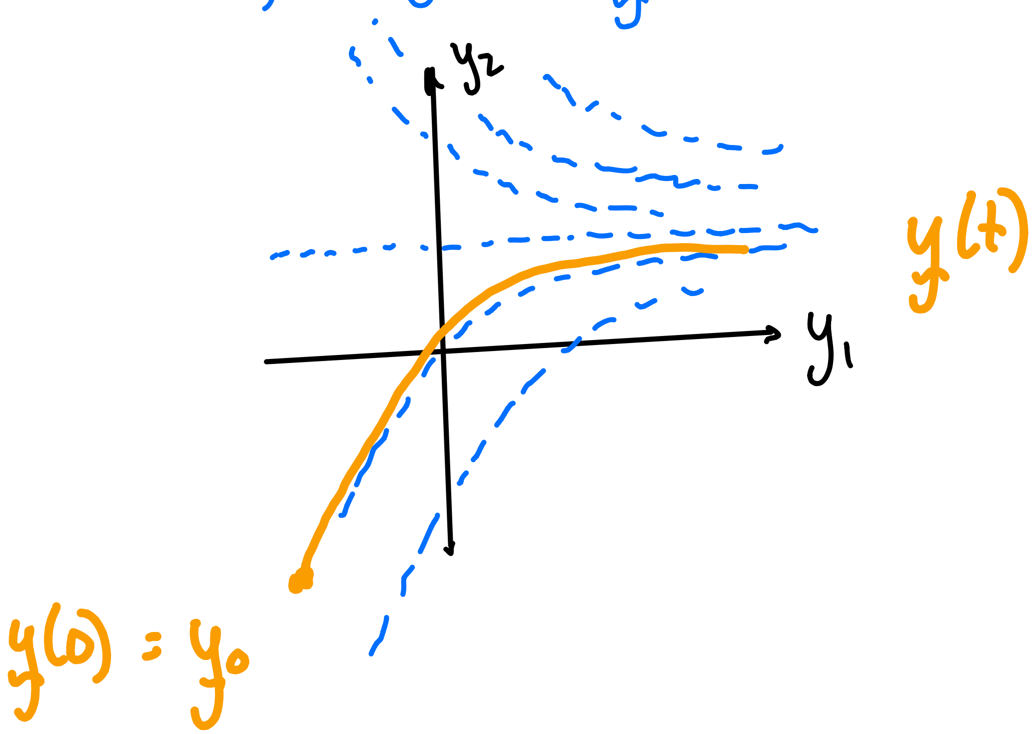
i.e. no explicit time dependence in $\underset{\sim}{f}$.

An autonomous system

$$\frac{dy}{dt} = \underset{\sim}{f}(y)$$

has derivatives that only depend on the state of y and not t explicitly.

Slope Fields: $\frac{dy}{dt} = f(y)$ indicates that the derivative of y does not change in time and that the **slope** of the solution $y(t)$ is given by $f(y)$. Plots of the slopes allow us to reconstruct y, since the derivative is always tangent to y:



The slope field is less useful in the case $\underset{\sim}{f}(y,t)$ since the slope field is changing in time....

CRITICAL (EQUILIBRIUM) POINTS

of a $\dfrac{dy}{dt} = \underset{\sim}{f}(y)$ are vectors y^* such that

$$\frac{dy^*}{dt} = 0 = f(y^*)$$

i.e. y^* are roots of $f(y) = 0$ If t is interpreted as time, then we also call the equilibrium points Stationary Points of $\dfrac{dy}{dt} = f(y)$.

ex) $$\frac{dx}{dt} = -(x-y)(1-x-y)$$

$$\frac{dy}{dt} = x(2+y) \qquad t>0$$

let $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ let $\underset{\sim}{f} = \begin{pmatrix} -(x-y)(1-x-y) \\ x(2+y) \end{pmatrix}$

then $$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}(\underset{\sim}{x}) \quad t>0$$

is an autonomous dynamical system.

To find critical points: set $\frac{d\underset{\sim}{x}}{dt} = 0$

and solve for x^*, y^* in

$$(x-y)(1-x-y) = 0$$

$$x(2+y) = 0$$

The critical pts are: $\underset{\sim}{x}_1^* = (0,0)$,

$\underset{\sim}{x}_2^* = (0,1)$, $\underset{\sim}{x}_3^* = (-2,-2)$, $x_4^* = (3,-2)$ //

ex) Find the critical points to

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x}$$

$$A \in \mathbb{R}^{n\times n} \quad \underset{\sim}{x} \in \mathbb{R}^n$$

here $\underset{\sim}{f} = A\underset{\sim}{x}$. ODE is autonomous.

The only critical point is $\underset{\sim}{x}^* = \underset{\sim}{0}$. //

STABILITY OF CRITICAL POINTS

For $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, $t>0$ with critical points

$\underset{\sim}{x}^*$, i.e. $\underset{\sim}{f}(\underset{\sim}{x}^*) = 0$,

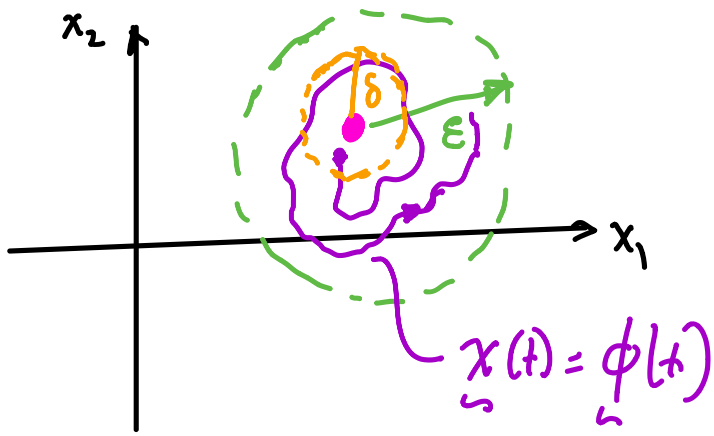
is said to be **STABLE** if

for every $\underset{\sim}{x} = \underset{\sim}{\phi}(t)$ which satisfies

$$\|\underset{\sim}{\phi}(0) - \underset{\sim}{x}^*\| < \delta \qquad \delta > 0$$

there exists

$$\|\underset{\sim}{\phi}(t) - x^*\| < \varepsilon \qquad \text{for } t \geq 0$$



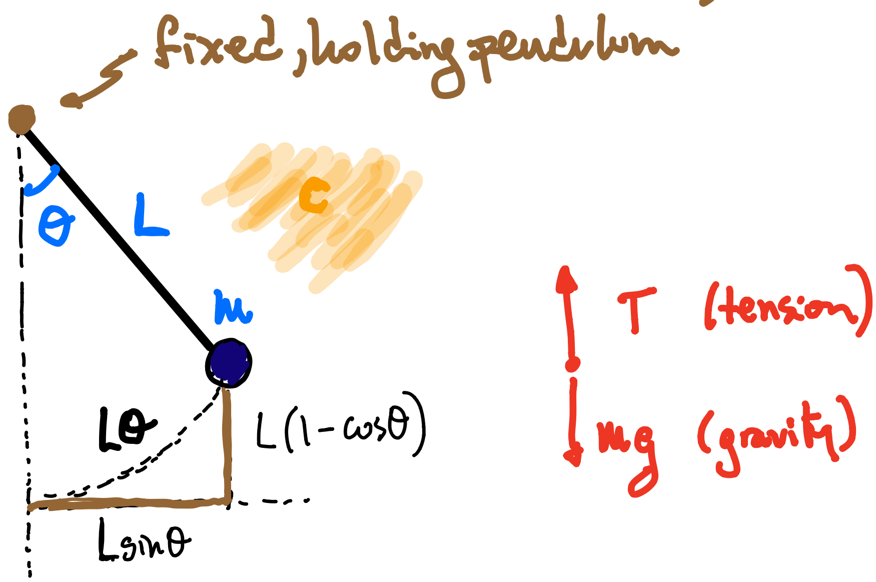
So $\underset{\sim}{\phi}(t)$ remains in some neighborhood of x^*.

A critical point is **ASYMPTOTICALLY** stable if it is stable and there exists a $\delta_0 > 0$ s.t.

$$\|\underset{\sim}{\phi}(0) - \underset{\sim}{x}^*\| < \delta_0 \text{ and } \lim_{t \to \infty} \underset{\sim}{\phi}(t) = \underset{\sim}{x}^*$$

i.e. it will reach a critical point as $t \to \infty$. //

THE PENDULUM (AN APPLICATION)



Momentum Conservation

$$(\star) \quad mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta$$

$c \geq 0$ but small Resistive Coefficient

$$[c] = \frac{ML}{T}$$

let $\begin{cases} \omega^2 \equiv g/L & [\omega] = T^{-1} \\ \gamma \equiv \dfrac{c}{mL} \geq 0 & [\gamma] = T^{-1} \end{cases}$

Substituting into (✱)

$$(\dagger) \qquad \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

let $x \equiv \theta$ $\qquad$ let $\underset{\sim}{X} = \begin{pmatrix} x \\ y \end{pmatrix}$

$y \equiv \frac{d\theta}{dt}$

$\therefore$ $(\dagger)$ is cast as

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

or $\frac{d\underset{\sim}{X}}{dt} = \underset{\sim}{f}(\underset{\sim}{X})$

where $\underset{\sim}{f}(\underset{\sim}{X}) = \begin{pmatrix} y \\ -\omega \sin x - \gamma y \end{pmatrix}$

Rmk: Clearly, an autonomous system

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}(\underset{\sim}{x}).$$

Find Critical Points: (x^*, y^*)

set $\quad \dfrac{d\underline{x}}{dt} = 0 = \begin{pmatrix} y^* \\ -\omega \sin x^* - \gamma y^* \end{pmatrix}$

we find a countably-infinite set:

$$y^* = 0 \qquad -\omega^2 \sin x^* - \gamma y^* = 0$$

or $\quad y^* = 0 \qquad x^* = \pm n\pi \quad n \in \mathbb{Z}$

Stability of Critical Points (c.p.)

Rmk: so clearly the c.p. correspond to the bob being either downright or upright:

$y^* = 0$

$x^* = n\pi$, n even

$y^* = 0$

$x^* = n\pi$ n odd

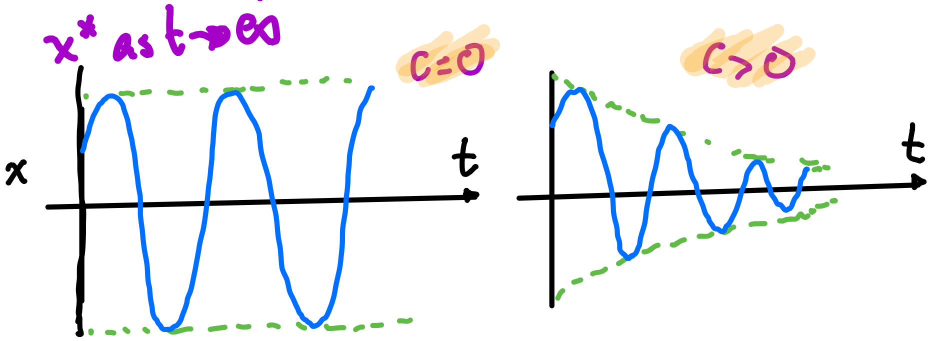
To investigate stability of both c.p.

let $x = x^* + \delta\theta$ (say, at $t=0$)

$|\delta\theta| \ll 1$

Rmk: What do we expect?

For n even: $x^*=0$ is stable ($c=0$) and asymptotically stable ($c>0$). $x(t)$ will always stay close to x^* or decay to x^* as $t\to\infty$



For n odd we expect that the slightest perturbation $\delta\theta$ will lead $x(t)$ to depart $x^*=\pi$. Unstable

Rmk: To investigate the stability we will examine how small perturbations $\delta\theta$ affect the behavior of $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ near (x^*, y^*) as $t\to\infty$

This is a (local) linear Stability Analysis

LOCAL LINEAR STABILITY ANALYSIS

We look at the dynamics of $\delta \underset{\sim}{x}$, small perturbations about $\underset{\sim}{x}^*$, where $|\underset{\sim}{x}| = O(1)$ and $|\delta \underset{\sim}{x}| \ll 1$.

$$(\ast) \qquad \frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}(\underset{\sim}{x}) \qquad \begin{array}{l} \underset{\sim}{x} \in \mathbb{R}^n \\ \underset{\sim}{f} \in \mathbb{R}^n \end{array}$$

Assume the existence of e.p. $\underset{\sim}{x}^*$

i.e. $\underset{\sim}{f}(\underset{\sim}{x}^*) = 0$

Let $\underset{\sim}{x}(t) = x^* + \delta x(t)$

Substitute into $(\ast)$:

$$\frac{d}{dt}(\underset{\sim}{x}^* + \delta \underset{\sim}{x}) = \underset{\sim}{f}(\underset{\sim}{x}^* + \delta \underset{\sim}{x})$$

but $\frac{dx^*}{dt} = 0$

$$\frac{d}{dt}\delta \underset{\sim}{x} = \underset{\sim}{f}(\underset{\sim}{x}^* + \delta \underset{\sim}{x})$$

$$= \underset{\sim}{f}(x^*) + \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}\bigg|_{\underset{\sim}{x} = x^*} \delta \underset{\sim}{x} + \frac{1}{2}\left[\frac{\partial^2 \underset{\sim}{f}}{\partial \underset{\sim}{x}^2}\right] \delta \underset{\sim}{x}^2 + \cdots$$

but $f(x^*) = 0$

and we're assuming that $|\delta x| \ll 1$

$$\therefore \quad \frac{d}{dt}\delta \underset{\sim}{x} = \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x}=\underset{\sim}{x}^*} \delta \underset{\sim}{x} + O(|\delta x|^2) \qquad (\#)$$

The Matrix

$$J \equiv \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x}=\underset{\sim}{x}^*} \in \mathbb{R}^{n\times n}$$ is called

the Jacobian matrix

Rmk: To calculate the Jacobian matrix, assume $\underset{\sim}{f} \in \mathbb{R}^n$ and $\underset{\sim}{x} \in \mathbb{R}^n$, $\underset{\sim}{f}$ continuous & $\underset{\sim}{f}'$ continuous, for $\underset{\sim}{x}$ near x^*.

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \qquad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then $J = \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x}=x^*}$ becomes a matrix of constants ex) let $\frac{d}{dt} \equiv (\dot{\ })$

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 \sin x - \gamma y$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\underset{\sim}{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 \sin x - \gamma y \end{pmatrix} \in \mathbb{R}^2$$

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -\gamma \end{pmatrix}$$

$$J = \left. \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} \right|_{x = x^*} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -\gamma \end{pmatrix}$$

For n even:

$$J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}$$

bob at $\left\{ \begin{array}{l} \theta^* = 0 \\ \frac{d\theta^*}{dt} = 0 \end{array} \right.$

For n odd:

$$J = \begin{pmatrix} 0 & 1 \\ +\omega^2 & -\gamma \end{pmatrix}$$

bob at $\left\{ \begin{array}{l} \theta^* = \pi \\ \frac{d\theta^*}{dt} = 0 \end{array} \right.$

//

Back to Stability

<u>Rmk</u>: Want to examine whether $\underset{\sim}{\delta x}$ perturbations about $\underset{\sim}{x}^*$ remain close to x^*, as $t \to \infty$.

<u>Rmk</u>: Linear (infinitesimal) perturbations will either oscillate, grow, decay, or a combination of oscillation/growth or oscillation/decay.

$$\frac{d}{dt}\underset{\sim}{\delta x} = J\,\underset{\sim}{\delta x}$$

<u>Rmk</u>: this is a system of the form
$$\dot{x} = Ax$$
so we know how to solve

The solution $\underset{\sim}{\delta x} = c_1 \underset{\sim}{v}_1 e^{\lambda_1 t} + c_2 \underset{\sim}{v}_2 e^{\lambda_2 t} + \dots + c_n \underset{\sim}{v}_n e^{\lambda_n t}$

Found by computing

$$\det(J - \lambda_i I) = 0 \qquad i = 1, 2, \dots, n$$

$$(J - \lambda_i I)\underset{\sim}{v}_i = 0$$

<u>Rmk</u>: You might need to use the reduction of order procedure if system has repeated λ.

__Rmk:__ When can you justify a linear approximation, in some radius $r = |\delta \underset{\sim}{x}|$ of $\underset{\sim}{x}^*$?

① We require that the $O(|\delta x|^2)$ terms remain small for all t, for δx sufficiently small:

i.e. $$\frac{d}{dt}(\underset{\sim}{x}^* + \delta \underset{\sim}{x}) = \frac{d}{dt}\delta \underset{\sim}{x} = J\,\delta \underset{\sim}{x} + O(|\delta \underset{\sim}{x}|^2)$$

small $\forall t$

Equation (#) is of the form

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{g}(\underset{\sim}{x})$$

where $\underset{\sim}{g}(\underset{\sim}{x}) = O(|\delta x|^2)$

Hence the requirement of $O(|\delta x|^2)$ be small means that

$$\lim_{\|\underset{\sim}{x}\| \to 0} \frac{\| \underset{\sim}{g}(\underset{\sim}{x}) \|}{\|\underset{\sim}{x}\|} = 0$$

This requirement should be true x near x^*, i.e.

$$\lim_{\|\delta x\| \to 0} \frac{\| g(x^* + \delta x) \|}{\|\delta x\|}$$

(2) We also require that $J\,\delta\underset{\sim}{x} = \underset{\sim}{0}$

i.e. that a linear approximation exist.

(we already assumed that $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}$ be continuous near $\underset{\sim}{x}^*$)

(3) We also require that $\underset{\sim}{x}_j^*$ be isolated

i.e. $\exists$ a ball around $\underset{\sim}{x}_j^*$ containing $\underset{\sim}{x}_j^*$ and no other critical $x^* \neq \underset{\sim}{x}_j^*$. //

RETURN TO PENDULUM PROBLEM:

Case $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$

$$\frac{d}{dt}\delta\underset{\sim}{x} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\delta\underset{\sim}{x} \equiv A\,\delta\underset{\sim}{x}$$

$$\det(A - \lambda I) = 0 = -\lambda(-\gamma - \lambda) + \omega^2 = 0$$

$$\lambda^2 + \gamma\lambda + \omega^2 = 0$$

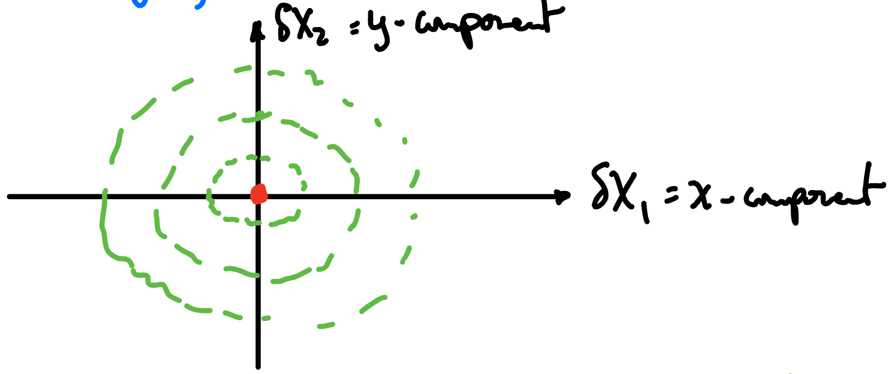
$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{i}{2}\sqrt{4\omega^2 - \gamma^2} \equiv -\frac{\gamma}{2} \pm i\omega_d$$

$$\delta\underset{\sim}{x} = e^{-\gamma/2\, t}\left[c_1 \underset{\sim}{v}_1 e^{i\omega_d t} + c_2 \underset{\sim}{v}_2 e^{-i\omega_d t}\right]$$

if $\gamma = 0$: $\lambda_{1,2} = \pm i\omega$

the solution $\delta \underset{\sim}{X}$ is oscillatory

then $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a CENTER (stable)



These are curves of constant energy. Perturbations oscillate forever.

if $\gamma > 0$: the solution has a time dependence $e^{-\frac{\gamma}{2}t} \left\{ \begin{matrix} \cos \omega_d t \\ \sin \omega_d t \end{matrix} \right\}$

where $\omega_d \equiv \{\omega^2 - (\gamma/2)^2\}$

the damped frequency.

So solutions decay as $t \to \infty$ but also oscillate if $\omega^2 > (\gamma/2)^2$ STABLE SPIRAL

or simply decay if $\omega^2 \leq (\gamma/2)^2$

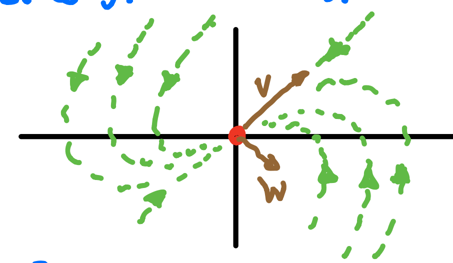
STABLE NODE

There are 2 cases: $4\omega^2 = \gamma^2$ and $4\omega^2 < \gamma^2$

if $\gamma^2 = 4\omega^2 \Rightarrow \lambda = -\gamma/2$

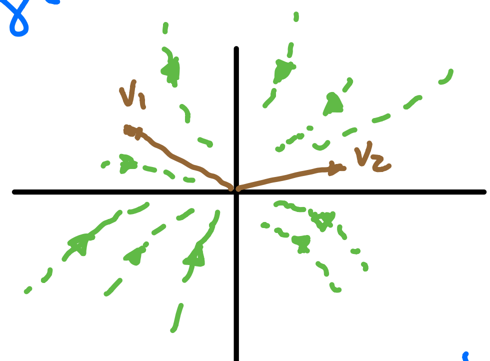
$$\delta \underset{\sim}{x} = c_1 \underset{\sim}{v} e^{\lambda t} + c_2 (\underset{\sim}{w} + \underset{\sim}{v} t) e^{\lambda t}$$

since $\lambda < 0$ the $\delta \underset{\sim}{x}$ decay to 0 as $t \to \infty$



Improper stable node

if $4\omega^2 < \gamma^2$



$$\delta \underset{\sim}{x} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = -\frac{\gamma}{2}(1 \mp \delta) \quad \text{where } \delta < 1$$

$$\therefore \lambda_{1,2} < 0$$

Case $x^* = \begin{pmatrix} \pi \\ 0 \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$

$$\frac{d}{dt} \delta \underset{\sim}{x} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \delta \underset{\sim}{x} = A \, \delta \underset{\sim}{x}$$

$$\det(A - \lambda I) = 0 = -\lambda(-\gamma - \lambda) - \omega^2 = 0$$

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2+4\omega^2}$$

One of the eigenvectors will has exponential growth,
the other has exponential decay.

A SADDLE (UNSTABLE)

$$\delta \underset{\sim}{x} = e^{-\gamma t/2}\left(c_1 \underset{\sim}{v}_1 e^{\frac{1}{2}\sqrt{\gamma^2+4\omega^2}\,t} + c_2 \underset{\sim}{v}_2 e^{-\frac{1}{2}\sqrt{\gamma^2+4\omega^2}\,t}\right)$$

clearly, as $t\to\infty$, the $\underset{\sim}{v}_2$ contribution becomes small compared to $\underset{\sim}{v}_1$ component.

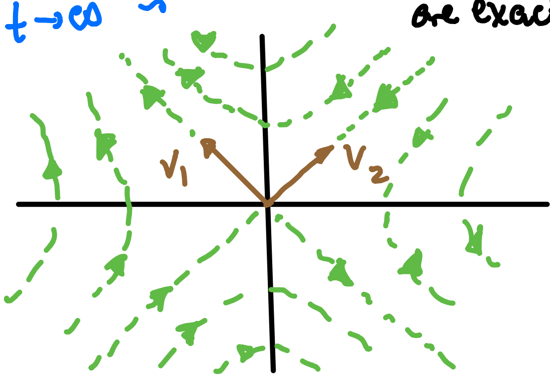
So for $t\to\infty$

$$\delta \underset{\sim}{x} \approx e^{-\gamma t/2} c_1 v_1 e^{\frac{1}{2}\sqrt{\gamma^2+4\omega^2}\,t}$$

$$= e^{\frac{\gamma}{2}t\left(\sqrt{1+\varepsilon}\,-1\right)} c_1 \underset{\sim}{v}_1$$

where $\varepsilon > 0$

$\therefore \lim_{t\to\infty} \delta \underset{\sim}{x}$ diverges (unless the dynamics are exactly aligned with $\underset{\sim}{v}_2$)



SADDLE
(UNSTABLE)

THE PHASE PLANE

These are plots of the position & momentum (or velocity) variables of the dynamics.

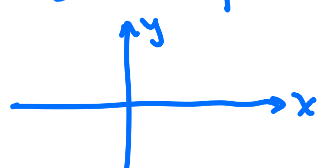
For the pendulum problem, the position $\theta = x$, and the velocity is $\frac{d\theta}{dt} = y$

To construct the phase plane for

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dx} = -\omega^2 \sin x - \gamma y \end{cases}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\omega^2 \sin x - \gamma y}{y}$$

We then produce a phase plane for $\frac{dy}{dx}$



Rmk: Let's build the phase plane for an easy case, when $\gamma = 0$:

$$\frac{dy}{dx} = \frac{y}{-\omega^2 \sin x}$$

$$\therefore\ -\omega^2 \sin x\, dx = y\, dy$$

Seperable ODE so we can integrate:

$$(\#)\quad C = \frac{1}{2}y^2 + \omega^2(1-\cos x)$$

where c is a constant

Multiply $(\#)$ by mL^2 and revert back to θ and $d\theta/dt$ variables:

$$mL^2 C = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

The units of $\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$ is energy.

$\therefore$ let $mL^2 c = E$

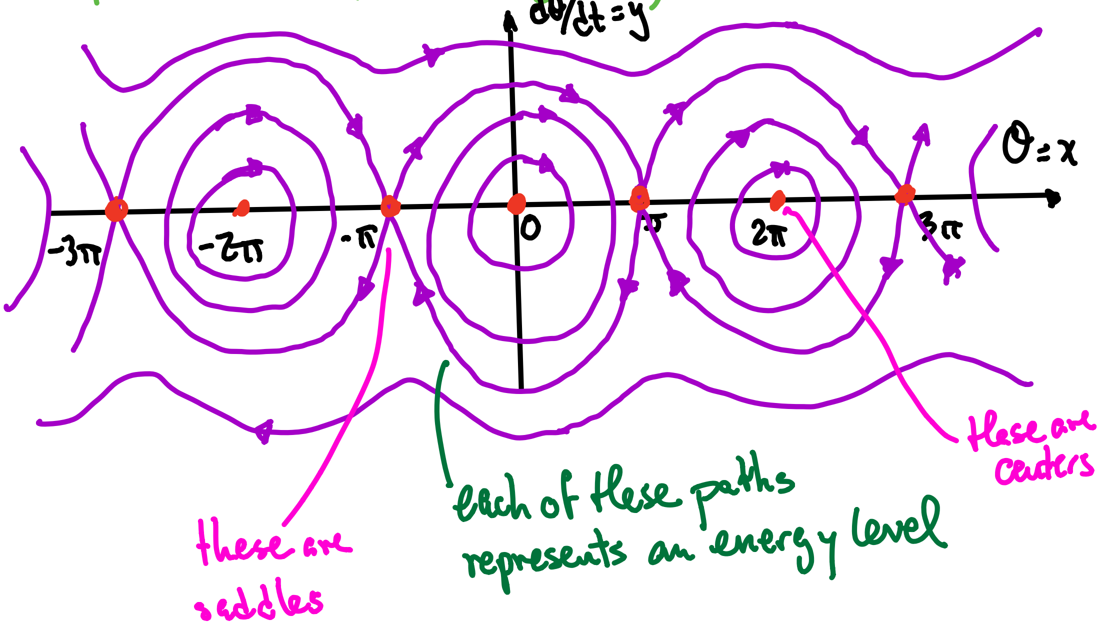
$$\Rightarrow \underbrace{E}_{\text{total energy}} = \underbrace{\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2}_{\text{Kinetic energy}} + \underbrace{mgL(1-\cos\theta)}_{\text{potential energy}} \qquad (\cancel{\theta})$$

We note that if $\frac{d\theta}{dt}(t=0) = A$ and $\theta(t=0) = B$ initial conditions then

$$E(t=0) = \frac{1}{2}mL^2A^2 + mgL(1-\cos B)$$
$$= E(t)$$

it's a constant of motion.

Phase Plane (Plotting) we see



<u>Rmk</u>: The above phase plane captures dynamics GLOBALLY. Certain system called "hyperbolic" can be fully characterized globally from a local analysis of critical points. //

Rmk: There are serious limitations to local linear stability. In what follows we consider one such situation. We focus on it because (a) it's easy, (b) common & possibly easily identified, (c) there are a few theoretical constructs that allow us to discern whether they are relevant to our analysis of a system for which little is known.

def: Periodic Solutions a periodic solution to $\dot{\underset{\sim}{x}} = \underline{f}(\underset{\sim}{x})$, $t>0$, satisfies

$$(*) \quad \underset{\sim}{x}(t+T) = \underset{\sim}{x}(t) \quad \forall t$$

where $T \in \mathbb{R}$ is called the period. It is the smallest number for which (*) is true. //

PERIODIC SOLUTIONS & LIMIT CYCLES

linear stability analysis has rather limited dynamics in time: things grow/decay exponentially or as a power of t, and they can also oscillate. It cannot capture a class of problems that are periodic called limit cycles.

let's look a system that has a limit cycle & show that linear stability analysis yields the wrong qualitative outcomes.

ex) Consider

$$(\ast) \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y-x(x^2+y^2) \\ -x+y-y(x^2+y^2) \end{pmatrix}$$

let $\underset{\sim}{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix}$.

$(\mp)$ has $\underset{\sim}{x}^* = (0,0)$ as a critical point.

Let's do linear stability analysis:

$$\underset{\sim}{x} = \underset{\sim}{x}^* + \delta \underset{\sim}{x}, \text{ then from } (\mp)$$

$$\frac{d}{dt}\delta \underset{\sim}{x} = J \delta \underset{\sim}{x} \quad \text{where } J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$p(\lambda) = \det(J - \lambda I) = 0$$

$$\lambda_{1,2} = 1 \pm i$$

So the $\delta \underset{\sim}{x}$ has exponentially growing oscillating solutions. We see that x^* should be unstable. Unstable spiral.

It turns out that we can solve $(\mp)$ exactly:

let $x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\tan\theta = y/x$

Substitute into $(\exists)$:

$$(\ddagger) \quad r\frac{dr}{dt} = r^2(1-r^2)$$

i.e. $$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2+y^2) - (x^2+y^2)^2$$

The solution of $(\ddagger)$ is $r(t) = 1$

To find an equation for θ: Multiply the first eq of $\exists$ by y and subtract the second multiplied by x:

$$y\frac{dx}{dt} - x\frac{dy}{dt} = x^2+y^2$$

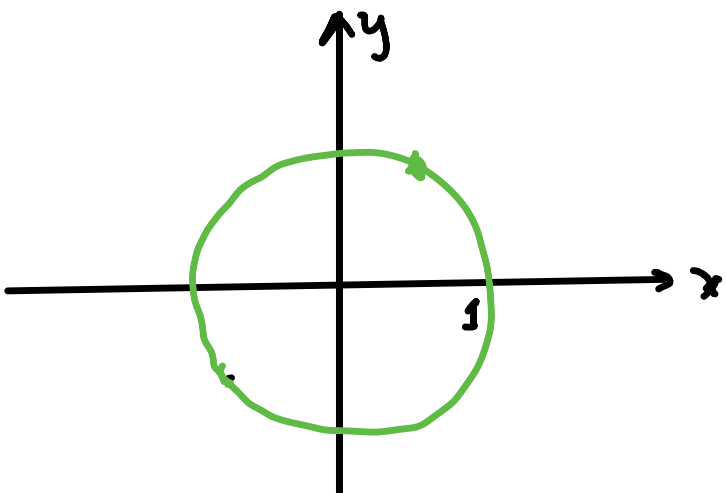
$$r\sin\theta\frac{d}{dt}(r\cos\theta) - r\cos\theta\frac{d}{dt}(r\sin\theta) = r^2$$

or $-r^2 \frac{d\theta}{dt} = r^2$

$\therefore \frac{d\theta}{dt} = -1 \Rightarrow \theta(t) = -t + \theta_0$

So the exact solution of $(\mp)$ is

$r = 1, \theta = -t + \theta_0$, and clearly periodic



Rmk: This is not the only solution.

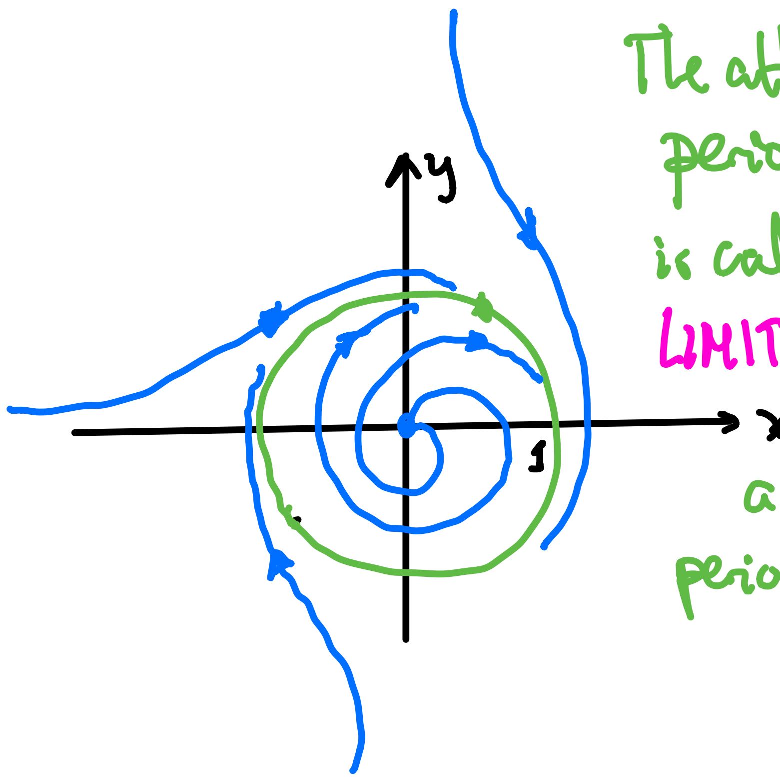
We write $(\mp)$ as

$\frac{dr}{r(1-r^2)} = dt$ partial fractions:

$$\left(\frac{1}{r} - \frac{1}{2}\frac{1}{1-r} + \frac{1}{2}\frac{1}{1+r}\right)dr = dt$$

Integrate b.s:

$$r(t) = \frac{1}{\sqrt{1+C_0e^{-2t}}}, \quad \theta(t) = -t + \theta_0$$



The attracting periodic orbit is called the LIMIT CYCLE (it is a STABLE periodic solution)

<u>def</u>: A trajectory that encloses other non-closed trajectories is called a LIMIT CYCLE //

It is asymptotically stable since as $t \to \infty$ all orbits tend to $r = 1$.

ex) Perform an analysis of the vander Pol Oscillator

$$\ddot{u} + u - \mu(1-u^2)\dot{u} = 0$$

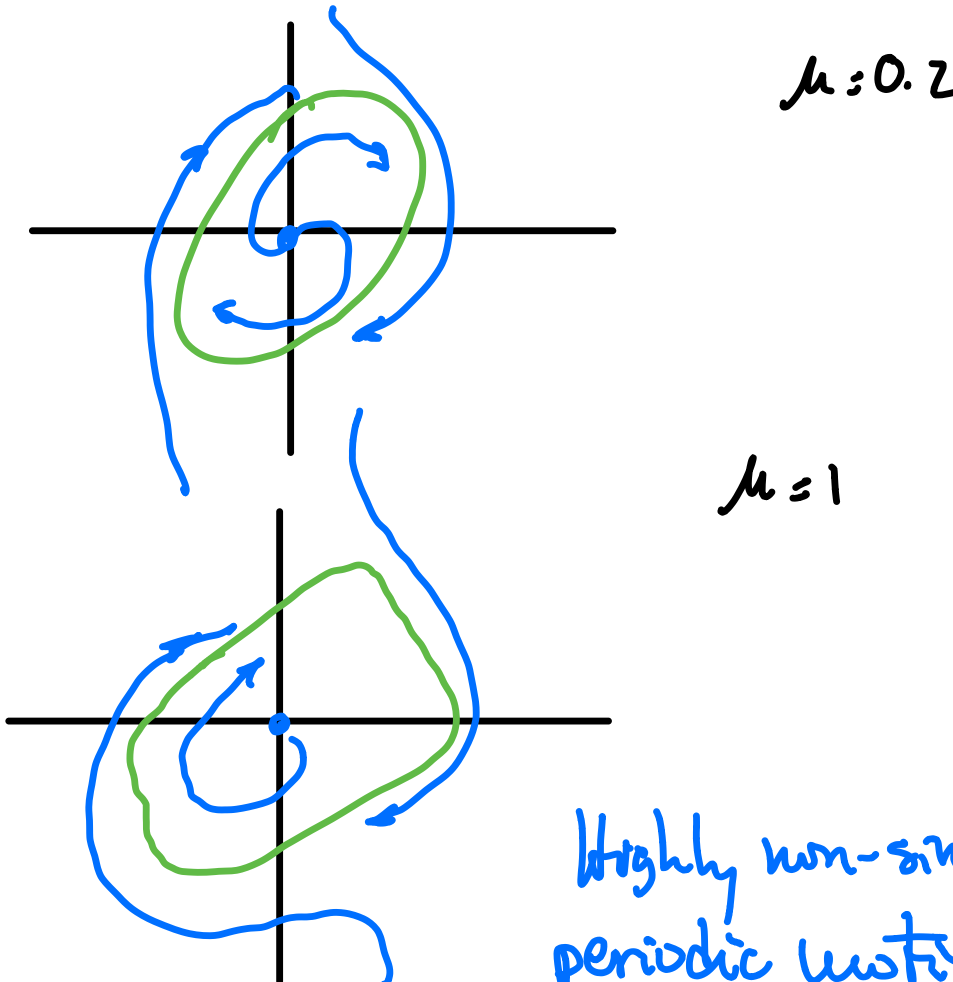
$$t > 0$$

$$\mu \geq 0$$

if $\mu = 0$ we get the sinusoidal functions, periodic with period 2π.

if $\mu > 0$: if u is large, the 3rd term is a damping term. If u is small, the 3rd term amplifies

Show that for $\mu \geq 0.2$ we get a limit cycle.



Highly non-sinusoidal periodic motion.

Look on the web page for the van der Pol oscillator demo.

Rmk: Clearly, it'd be good to know when we expect the linear analysis to work. We can, for periodic cases:

THREE THEOREMS FOR 2D PROBLEMS

Assume
$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \qquad (\$)$$

Thm F & G continuous with continuous first partials in some domain $(x,y) \in D$ of the x,y plane. A closed trajectory of $(\$)$ must necessarily enclose at least 1 critical point. If it encloses a single critical point it cannot be a (unstable) saddle. //

Thm F & G as above. If
$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \equiv \nabla \cdot (F,G)$$
in a simply-connected domain D of xy plane has the same sign throughout D

$\Rightarrow$ there's no closed trajectory of ($) lying entirely in D //

Rmk: A simply-connected region has no holes.

Thm Poincaré Bendixson Theorem: let $R = D \cap \partial D$

be a region with continuous F & G & continuous partials.

If there is a periodic solution that stays inside R (closed orbit) OR there's a solution that spirals toward a closed orbit as $t \to \infty$

The system has a periodic orbit in R (a limit cycle). //

BENDIXSON'S CRITERION: If $\frac{\partial F}{\partial x}$ and $\frac{\partial G}{\partial y}$ are continuous in R, a simply-connected region and

$$\nabla \cdot (F, G) \neq 0 \text{ for any point in } R$$

then $\dot{x} = F \quad \dot{y} = G$

has no closed trajectories in R

CRITICAL POINT CRITERION: A closed trajectory has a critical point in its interior.

ex) For what a, d does

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

have closed trajectories?

if $a+d=0$ Poincaré Bendixson says nothing

look at e'values:

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

* Roots are complex if $ad-bc>0$
  system will have a center if $a+d=0$
  this is the closed trajectories case.

* if $ad-bc<0$ and $a+d=0$ the system is a saddle
  non-closed trajectories.