STABILITY PROBLEMS WITH PARAMETERS: BIFURCATIONS

Taken mostly from J.D. Logan's Book.

<u>Rmk</u>: We'll focus on 1D Problems

Consider

$$(\star) \quad \frac{du}{dt} = f(u,\mu) \quad t>0$$

μ is a real parameter

If $(\star)$ has a c.p. (critical point) u^*

$$f(u^*,\mu) = 0$$

then $u^* = u^*(\mu)$, the c.p. is a 1-parameter family in μ.

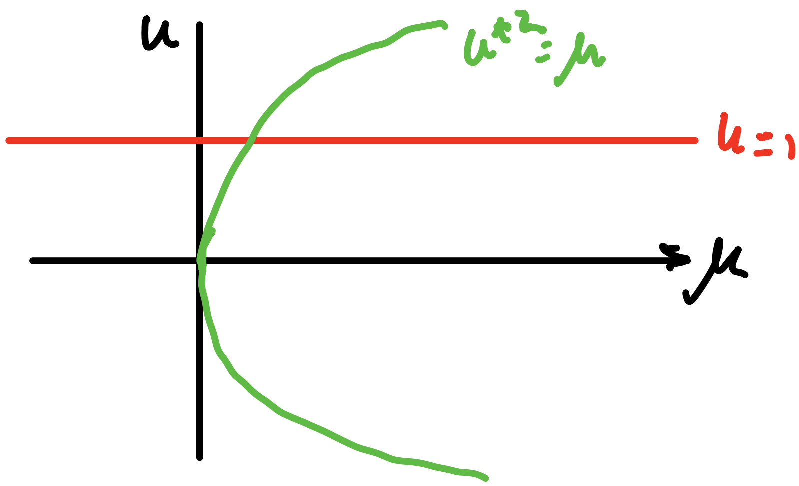
The graph of $u^*(\mu)$ in the (u,μ) plane is called the branching or bifurcation diagram.

ex) $\dfrac{du}{dt} = (1-u)(u^2-\mu) \qquad \mu \in \mathbb{R}^1$

$t > 0$

c.p. : $(1-u^*)(u^{*2}-\mu) = 0$

or $u^* = 1$ and $\mu = u^{*2}$



Rmk: no equilibrium exists for $\mu < 0$ in this example.

For a given fixed μ, a solution u^* is stable if, for every $u(t)$ solution of (*) starting sufficiently close to u^* at $t=0$, remains close to u^* for all $t>0$:

So $|u(t)-u^*|<\varepsilon \quad t>0$

whenever $|u(0)-u^*|<\delta$

If, in addition,

$$\lim_{t\to\infty} |u(t)-u^*| = 0$$

then u^* is asymptotically stable.

What conditions are required on (*) to lead to stable/asymptotically stable solutions?

let $u(t) = u^* + \delta u$ $\quad$ u^* is c.p.

replace into ($\star$)

$$\frac{d}{dt}\delta u = f(u^* + \delta u, \mu) \quad t>0$$

$$\frac{d\delta u}{dt} = f(u^*, \mu) + \frac{\partial f}{\partial u}(u^*, \mu)\delta u + O(\delta u^2)$$

if $|\delta u| << 1$

$$\frac{d\delta u}{dt} \approx \frac{\partial f}{\partial u}(u^*, \mu)\delta u$$

so $\delta u = Ce^{\alpha t}$

when $\alpha = \frac{\partial f}{\partial u}(u^*, \mu)$

if $\alpha < 0 \Rightarrow \delta u \to 0$ as $t \to \infty$

$\alpha > 0 \Rightarrow \delta u \to \infty$ as $t \to \infty$

$\alpha = 0$, nothing can be said of analysis.

A useful tool for stability theory is Gronwall's Inequality.

Lemma If $u(t)$ and $v(t)$ are continuous non-negative functions on $0 \le t \le T$ and K is non-negative constant then

$$(*) \quad u(t) \le K + \int_0^t v(s)u(s)\,ds \qquad 0 \le t \le T$$

implies

$$u(t) \le K e^{\int_0^t v(s)\,ds}$$

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Pf: if $K > 0 \Rightarrow (*)$ says that

$$\frac{u(t)\,v(t)}{K + \int_0^t v(s)u(s)\,ds} \le v(t)$$

Integrating b.s.

$$\ln\left[K + \int_0^t v(s)u(s)\,ds\right] - \ln K \le \int_0^t v(s)\,ds$$

$$\therefore u(t) \le K + \int_0^t v(s)u(s)\,ds \le K \exp\left[\int_0^t v(s)\,ds\right].$$

If $K = 0 \Rightarrow u(t)$ is identically zero.

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Thm: Let u^* be a c.p. of $(\maltese)$. Assume

$$f(u^*+\delta u;\mu) = \frac{\partial f}{\partial u}(u^*;\mu)\,\delta u + R(u^*;\delta u)$$

where $R(u^*;\delta u) = O(\delta u^2)$. That is,

$$|R(u^*;\delta u)| \le K|\delta u|^2, \text{ for } |\delta u| \ll 1,\ K>0.$$

Then u^* is asymptotically stable if $\alpha<0$ and unstable if $\alpha>0$, where

$$\alpha = \left.\frac{\partial f}{\partial u}\right|_{u=u^*;\mu} \qquad //$$

Pf: $$\frac{d}{dt}\delta u = \frac{\partial f}{\partial u}(u^*;\mu)\,\delta u + R(u^*;\delta u) \qquad (\ast)$$

Multiply b.s. of $(\ast)$ by $e^{-\alpha t}$ and integrate in t:

$$\delta u(t) - \delta u(0)e^{\alpha t} = \int_0^t R(u^*;\delta u(s))\,e^{\alpha(t-s)}\,ds$$

$$\therefore\ |\delta u(t) - \delta u\, e^{\alpha t}| \le \int_0^t |R(u^*;\delta u(s))|\,e^{\alpha(t-s)}\,ds$$

$$\Rightarrow |\delta u(t) - \delta u(0) e^{\alpha t}| \leq K \int_0^T |\delta u(s)|^2 e^{\alpha(t-s)} ds$$

or

$$e^{-\alpha t} |\delta u(t)| \leq |\delta u(0)| + K \int_0^t |\delta u(s)|^2 e^{-\alpha s} ds.$$

$$\therefore |\delta u(t)| \leq |\delta u(0)| \exp\left[\alpha t + K \int_0^t |\delta u(s)| ds\right]$$

$$(\$)$$

Let $\alpha < 0$ and assume

$$(\exists) \quad |\delta u| \leq \frac{\eta}{K} \quad 0 \leq t \leq T$$

where η is chosen so that $\alpha + \eta < 0$, $T > 0$.

Then $(\$)$ implies

$$(\yen) \quad |\delta u(t)| \leq |\delta u(0)| e^{(\alpha+\eta)t} \quad 0 \leq t \leq T$$

$\therefore$ if $(\exists)$ holds then $(\yen)$ follows for $T > 0$.

But $(\exists)$ holds if $|\delta u(0)|$ is sufficiently small.

The above shows that for $\alpha < 0$ & $|\delta u(0)|$ sufficiently small, u^* is asymptotically stable. //

ex) $\frac{du}{dt} = \mu u - u^2 = u(\mu - u) = f(u,\mu)$

so $u^* = 0$ and $\mu = u^*$ are c.p.

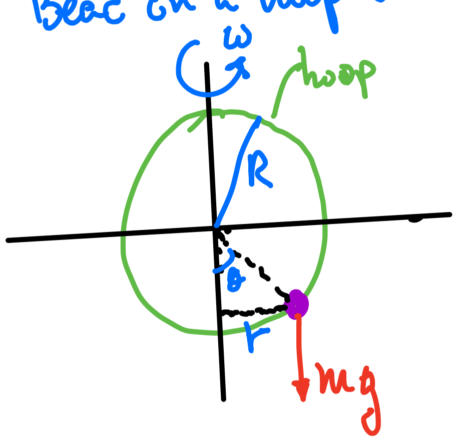
$\frac{\partial f}{\partial u} = \mu - 2u$ and $\alpha = \frac{\partial f}{\partial u}\Big|_{u^*}$

if $u^* = 0$ $\alpha = \mu$ and it is stable if $\mu < 0$.
if $u^* = \mu$ $\alpha = -\mu$ and it is stable if $\mu > 0$. //

CLASSIFICATION OF BIFURCATION POINTS

A system can change from stable to unstable & viceversa as a parameter changes.

ex) Bead on a hoop under gravity: the hoop is rotated with angular velocity ω, constant.



The bead's acceleration is $R\ddot{\theta}$

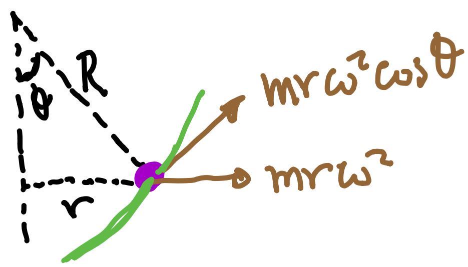
The tangential component of gravity force mg : $mg\sin\theta$

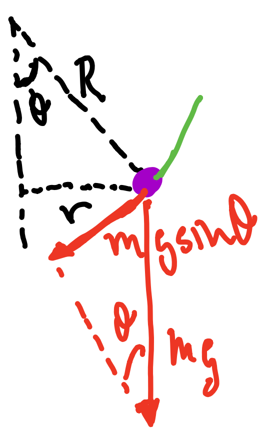
The centripetal force $mr\omega^2$ with tangential component $mr\omega^2\cos\theta$

$$mR\ddot{\theta} = -mg\sin\theta + mr\omega^2\cos\theta$$

since $r = R\sin\theta$

$$mR\ddot{\theta} = -mg\sin\theta + mR\omega^2\cos\theta\sin\theta$$





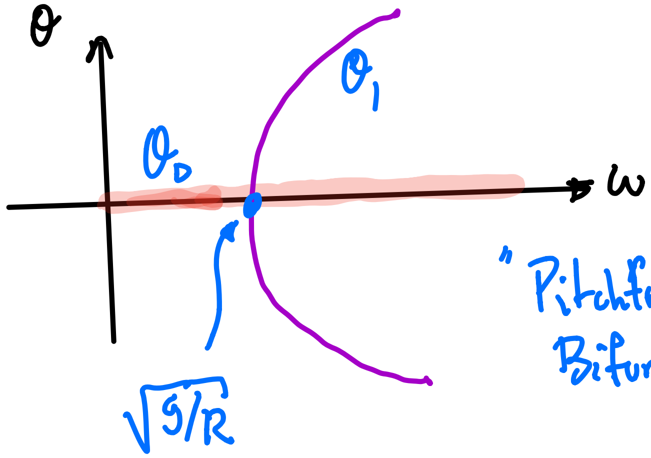
The equilibrium is reached when the acceleration is zero.

$$mR\omega^2\cos\theta\sin\theta = mg\sin\theta$$

or $R\omega^2 m\sin\theta\left(\cos\theta - \dfrac{g}{R\omega^2}\right) = 0$

The c.p. are:

$$\theta_0 = 0 \quad \text{and} \quad \theta_1 = \arccos\left(\frac{g}{R\omega^2}\right)$$



"Pitchfork Bifurcation"

For $0 \le \omega \le \sqrt{g/R}$ there's 1 equilibrium, $\theta_0 = 0$

For $\omega \ge \sqrt{g/R}$ there are 3 possible $\begin{cases} \theta_0 = 0 \\ \theta_1 = +\left|\arccos g/R\omega^2\right| \\ \theta_2 = -\left|\arccos g/R\omega^2\right| \end{cases}$

In the HW you get to analyze these and determine the stability

CLASSIFICATION & STRUCTURE OF ROOTS OF $f(u,\mu)=0$

Thm let $f(u,\mu)$ be a continuously differentiable function in some open region U of the

(u,μ) plane containing (u_0,μ_0)

If $\frac{\partial f}{\partial u}(u_0,\mu_0) \neq 0 \Rightarrow \exists$ a rectangle S

$S: |u-u_0|<a\ ,\ |\mu-\mu_0|<b$

inside U s.t.

(i) $f(u,\mu)=0$ has a unique solution

$u=u(\mu)$ on S.

(ii) $u(\mu)$ is continuously differentiable on $|\mu-\mu_0|<b$

and its derivative is given by

$$\frac{du}{d\mu} = -\frac{\partial f/\partial \mu}{\partial f/\partial u} = -\frac{f_\mu}{f_u}$$

Rmk: If $\mu=\mu(u)$ then $\frac{d\mu}{du} = -\frac{f_u}{f_\mu}$ //

REGULAR & SINGULAR POINTS

Let $P_0=(u_0,\mu_0)$

P_0 is regular if $f(P_0)=0$

and $\begin{cases} f_u(P_0) \neq 0 \\ f_\mu(P_0) \neq 0 \end{cases}$

then there's a unique curve

$u = u(\mu)$ passing through P_0

P_0 is SINGULAR if $f(P_0) = 0$

and $f_u(P_0) = f_\mu(P_0) = 0$

and $\frac{d\mu}{du}$ is indeterminate at P_0. //

S'pose that $f_{uu}, f_{u\mu}, f_{\mu\mu}$ do not all vanish simultaneously and f is continuous as are its first, second, third derivatives. Then

$$f(P) = f(P_0) + f_u(P_0)\Delta u + f_\mu(P_0)\Delta\mu$$
$$+ \frac{1}{2}\left[f_{uu}(P_0)\Delta u^2 + 2 f_{u\mu}(P_0)\Delta u \Delta\mu + f_{\mu\mu}\Delta\mu^2\right]$$
$$+ O(\Delta u^2) + O(\Delta\mu^2)$$

where $\Delta u = u - u_0$, $\Delta\mu = \mu - \mu_0$.

If P_0 is singular then

$$f(P_0) \overset{=0}{} = f(P_0)\overset{=0}{} + f_u(P_0)\overset{=0}{}\Delta u$$
$$+ f_\mu(P_0)\Delta\mu\overset{=0}{}$$
$$+ \frac{1}{2}\left[f_{uu}\Delta u^2 + 2 f_{u\mu}\Delta u\Delta\mu + f_{\mu\mu}\Delta\mu^2\right] +$$

$$O(\Delta u^3) + O(\Delta \mu^3)$$

Take $\Delta u \to 0$ $\Delta \mu \to 0$

and

$$f_{uu} \Delta u^2 + 2 f_{u\mu} \Delta u \Delta \mu + f_{\mu\mu} \Delta \mu^2 \Big|_{P_0} = 0$$

gives a relationship between $\Delta u, \Delta \mu$:

Let $D = f_{u\mu}^2(P_0) - f_{uu}(P_0) f_{\mu\mu}(P_0)$

the discriminant

$$\therefore \frac{du}{d\mu} = -\frac{f_{\mu u}}{f_{uu}} \pm \sqrt{\frac{D}{f_{uu}^2}}$$

or

$$\frac{d\mu}{du} = -\frac{f_{\mu u}}{f_{\mu\mu}} \pm \sqrt{\frac{D}{f_{\mu\mu}^2}}$$

Gives slopes of the tangents to the bifurcation curves near P_0.

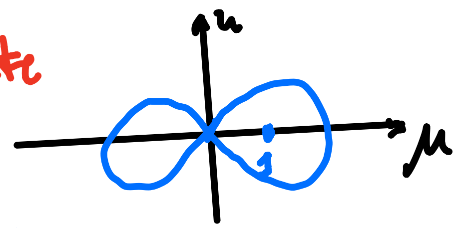
If $D > 0 \Rightarrow P_0$ is a double point with defined tangents

If $D<0 \Rightarrow P_0$ is isolated point with no real tangents

If $D=0 \Rightarrow$ at least 2 curves passing through P_0 have coincident tangents

ex) $f(u,\mu) = (\mu^2+u^2)^2 - 2(\mu^2-u^2) = 0$

the lemniscate



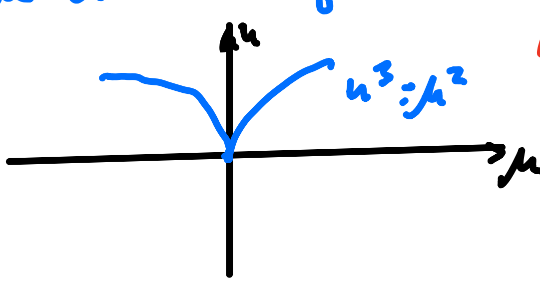
Check $f(0,0) = f_\mu(0,0) = f_u(0,0) = 0$

and $D(0,0) > 0 \therefore$

$(0,0)$ is a double point with 2 distinct tangents.

ex) $f(u,\mu) = u^3 - \mu^2 = 0$ has $(0,0)$ as a singular point with $D=0$. The 2 branches have coincident tangents.

the cusp



ex) $f(u,\mu) = u^2 + \mu^2 = 0$ $(0,0)$ is an isolated point with $D < 0$.

<u>Lemma</u>: Let P_0 be a double point of $f(u,\mu) = 0$ then either

(i) $f_{\mu\mu}(P_0) \neq 0$ and the 2 tangents are given by

$$\frac{d\mu}{du} = -\frac{f_{u\mu}}{f_{\mu\mu}} \pm \sqrt{\frac{D}{f_{\mu\mu}^2}}$$

or

(ii) $f_{\mu\mu}(P_0) = 0$ and the 2 tangents are given by

$$\frac{du}{d\mu} = 0 \quad \text{and} \quad \frac{d\mu}{du} = -\frac{f_{uu}}{2 f_{\mu u}} \quad \text{at } P = P_0$$

$\therefore$ there are 2 branches with distinct tangents pass through a double point P_0 and no more than 2 branches pass through a double point.

The following theorem states that if P_0 is a regular point the stability must change at a turning point.

Thm: Let P_0 be a regular point of $f(\mu, u) = 0$. Then equilibrium solutions on one side are stable and the other unstable:

Since $f_u = -\frac{d\mu}{du} f_\mu$, then

$$\alpha(u) = -\frac{d\mu}{du} f_\mu \Big|_{u=u^*}$$

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EXCHANGE OF STABILITY

2 branches cross double points. The question is whether the stability changes. But stability might also change at regular points

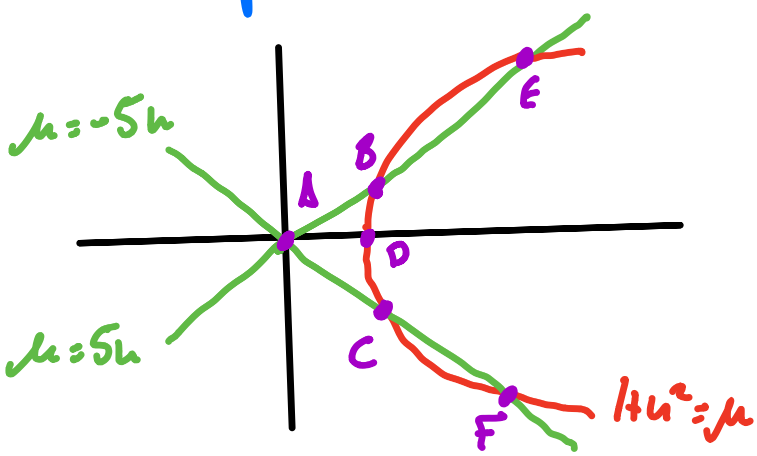
Thm: Let P_0 be regular point of $f(u,\mu) = 0$.

If $\frac{\partial f}{\partial \mu}(P_0) \neq 0$ and $\frac{d\mu}{du}$ changes sign at P_0, then P_0 is a regular turning point

On the other side of a turning point the equilibrium is stable & on the other, unstable. //

ex) $f(u,\mu) = (1+u^2-\mu)(\mu^2-25u^2)$

has c.p. $\mu_1 = 1+u^2$, $\mu_2 = 5u$, $\mu_3 = -5u$



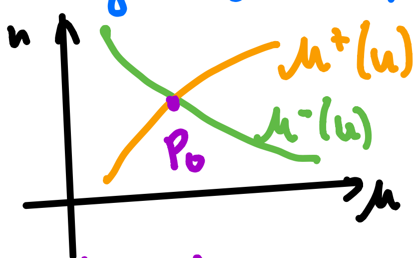
All points on the branches are regular except at A, B, C, E, F. The point D is regular, since $\frac{d\mu}{du}$ changes sign at D. The points A, B, C, E, F are double because they are singular points at which there are 2 distinct tangents.

EXCHANGE OF STABILITY AT DOUBLE POINT

There are 2 cases to consider (from lemma)

case (i) The 2 curves $\mu^+(u)$ and $\mu^-(u)$ passing through P_0 have tangents given by

$$\frac{d\mu}{du} = -\frac{f_{uu}}{f_{\mu\mu}} \pm \sqrt{\frac{D}{f_{\mu\mu}^2}}$$



We're trying to determine the stability indicators α^+, α^- associated with μ^+ and μ^-

If P_0 is a double point with $f_{\mu\mu}(P_0) \neq 0$

then

$$\alpha^+(u) = -\frac{d\mu^+}{du}(u) \operatorname{sgn} f_{\mu\mu}(P_0) \sqrt{D}\,(u-u_0)$$

$$\alpha^-(u) = \frac{d\mu^-}{du}(u) \operatorname{sgn} f_{\mu\mu}(P_0) \sqrt{D}\,(u-u_0)$$

$$D = f_{u\mu}^2(P_0) - f_{uu}(P_0)\, f_{\mu\mu}(P_0)$$

$\therefore$ Suppose $|u-u_0| \ll 1$ α^+, α^- have the same sign if $\frac{d\mu^+}{du}$ and $\frac{d\mu^-}{du}$ have opposite signs. They have opposite signs if $\frac{d\mu^+}{du}$ & $\frac{d\mu^-}{du}$ have the

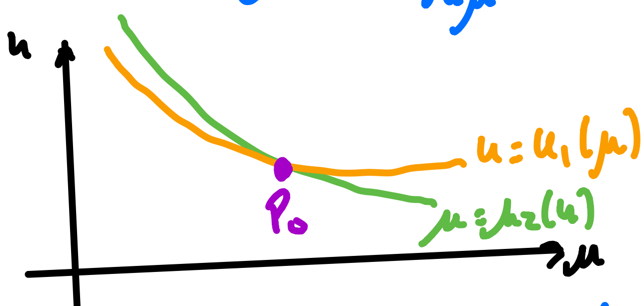
Same sign.

Case (ii) One curve $u = u_1(\mu)$ with

$$\frac{du_1}{d\mu} = 0 \text{ at } P_0 \ \&$$

the other $\mu = \mu_2(u)$ with

$$\frac{d\mu_2}{du} = -\frac{f_{uu}}{2f_{u\mu}} \text{ at } P_0$$



The stability of u_1 & μ_2 are given by

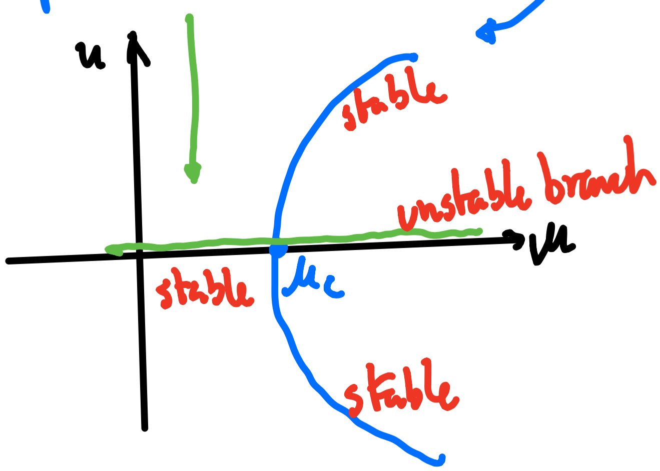
$$\alpha^1(\mu) = \frac{\partial f}{\partial u}(\mu, u_1(\mu)) = \operatorname{sgn} f_{u\mu}(P_0)\sqrt{D}(\mu - \mu_0)$$

$$\alpha^2(\mu) = -\operatorname{sgn} f_{u\mu}(P_0)\frac{d\mu_2}{du}(u)\sqrt{D}(u - u_0)$$

ex) let $\frac{du}{dt} = (\mu - \mu_c)u - u^2$

$\mu_c > 0$

c.p. $u_1 = 0$ $\quad \mu_2 = u^2 + \mu_c$



$P_0 = (\mu_c, 0)$ and clearly a double point.

$\frac{\partial f}{\partial u} = \mu - \mu_c - 3u^2 \qquad \frac{\partial f}{\partial \mu} = u$

$\frac{\partial^2 f}{\partial u^2} = -6u \qquad \frac{\partial^2 f}{\partial \mu^2} = 0 \qquad \frac{\partial^2 f}{\partial u \partial \mu} = 1$

Clearly, $f_\mu = f_u = 0$ at P_0

$f_{uu} = f_{\mu\mu} = 0$ at P_0 but

$f_{\mu u} \neq 0$ at P_0. The $D = 1$.

By lemma 2 distinct tangents are given by $\frac{du_1}{d\mu} = 0$ $\frac{d\mu_2}{du} = -\frac{f_{uu}}{2f_{\mu u}} = 0$

hence

$$\alpha^1(\mu) \approx \operatorname{sgn}(1)\sqrt{1}\,(\mu - \mu_0) = (\mu - \mu_c)$$

$$\alpha^2(u) \approx -\operatorname{sgn}(1)\,\frac{d}{du}(u^2 + \mu_c)\sqrt{1}\,(u - 0) = -2u^2$$

Since $\alpha^2(u) < 0$ $\mu_2 = u^2 + \mu_c$ is stable

$\alpha^1(u) > 0$ for $\mu > \mu_c$ $u_1 = 0$ is stable for $\mu < \mu_c$ and unstable for $\mu > \mu_c$

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