BVP Boundary Value Problems

- Homogeneous
- Nonhomogeneous

Sturm-Liouville Theory

Notes based on Boyce-DiPrima Book

<u>Rmk:</u> The key concepts are: 1 for BVP problems with mild conditions, a non-trivial solution to the homogeneous problem is possible. 2 Certain linear BVP have a unique solution. 3 BVP that are linear and of the Sturm-Liouville type can be spanned by a set of (possibly infinite) set of functions, which can be used to express any solution to the BVP as a linear superposition of the spanning, linearly independent, basis set.

LINEAR BVP

We'll focus on 2nd order ODE's, but a lot of this generalizes to higher order ODE's (in particular, to even order ODE's).

Consider $y = y(x)$ s.t.

$$\text{BVP} \begin{cases} (\text{ODE}) & y'' + p(x)y' + q(x)y = 0 \quad 0<x<1 \\ (\text{BC}) & \begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases} \end{cases}$$

We know such (ODE) has a ! (unique) solution provided $p(x)$ & $q(x)$ are continuous on $x \in [0,1]$.

However we cannot expect a ! solution to ODE + BC. First of, ODE + B.C admit a trivial solution $y = 0$, and could also admit $y = \phi(x) \neq 0$ as well as $y = k\phi(x)$ where k is constant $\therefore$ could admit an infinite set of solutions.

For ODE + B.C. could expect

$y=0$ trivial

$y=k\phi(x)$ single infinite set of solutions

$y=c_1 y_1(x)+c_2 y_2(x)$ doubly-infinite set of solutions

no solution

ex) Find $y=\phi(x)$ s.t.

ODE $y''+\pi^2 y=0$

B.C. $y(0)=0=y(1)$

$y=c_1\sin\pi x+c_2\cos\pi x$

solves ODE. Apply B.C.

$\therefore \begin{cases} c_2=0 \\ c_1 \text{ arbitrary} \end{cases} \Rightarrow y=c_1\sin\pi x.$

ex) Find $y=\phi(x)$ s.t.

$y''+\pi^2 y=0 \Rightarrow y=c_1\sin(\pi x)+c_2\cos(\pi x)$

$y(0)+y(1)=0,\ y'(0)+y'(1)=0$

Both satisfied with no restrictions on c_1 & c_2

$\therefore\ y = c_1 \sin \pi x + c_2 \cos \pi x$

two family, infinite set of solutions

ex) ODE: $y'' + \lambda^2 y = 0 \qquad \lambda \in \mathbb{R}$

B.C.: $y(0) = y(1) = 0$

$\therefore\ y = c_1 \sin n\pi x \qquad n = 1, 2, \ldots$

($n = 0$ is the trivial solution)

single family of solutions (infinite)

EIGENFUNCTIONS & EIGENVALUES

Consider

ODE $\quad y'' + p(x,\lambda)y' + q(x,\lambda)y = 0, \quad 0<x<1$

p & q continuous for $x \in [0,1]$

$\lambda \in \mathbb{R}$, a parameter

$$y(x) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$$

S'pse B.C.

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$$

$\therefore$

$$\alpha_1\left[c_1 y_1(0,\lambda)+c_2 y_2(0,\lambda)\right]+\alpha_2\left[c_1 y_1'(0,\lambda)+c_2 y_2'(0,\lambda)\right]=0$$

$$\beta_1\left[c_1 y_1(1,\lambda)+c_2 y_2(1,\lambda)\right]+\beta_2\left[c_1 y_1'(1,\lambda)+c_2 y_2'(1,\lambda)\right]=0$$

or

$$\left[\alpha_1 y_1(0,\lambda)+\alpha_2 y_1'(0,\lambda)\right]c_1+\left[\alpha_1 y_2(0,\lambda)+\alpha_2 y_2'(0,\lambda)\right]c_2=0$$

$$\left[\beta_1 y_1(1,\lambda)+\beta_2 y_1'(1,\lambda)\right]c_1+\left[\beta_1 y_2(1,\lambda)+\beta_2 y_2'(1,\lambda)\right]c_2=0$$

we will write as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=0$$

Is a homogeneous set of equations for the unknowns c_1 & c_2, other than zero, iff

$$(*) \quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=0$$

Since $a_{ij}=a_{ij}(\lambda)$

then $(*)$ can be used to find a λ or set of λ that lead to $c_1\neq 0$ and/or $c_2\neq 0$.

Consider $y'' + \alpha y = 0, \; 0<x<1$ ODE

$$\left.\begin{aligned} y(0) &= 0 \\ y(1) + y'(1) &= 0 \end{aligned}\right\} \text{B.C.}$$

$\boxed{\text{let } \alpha = \lambda^2}$

Consider $\alpha > 0$:

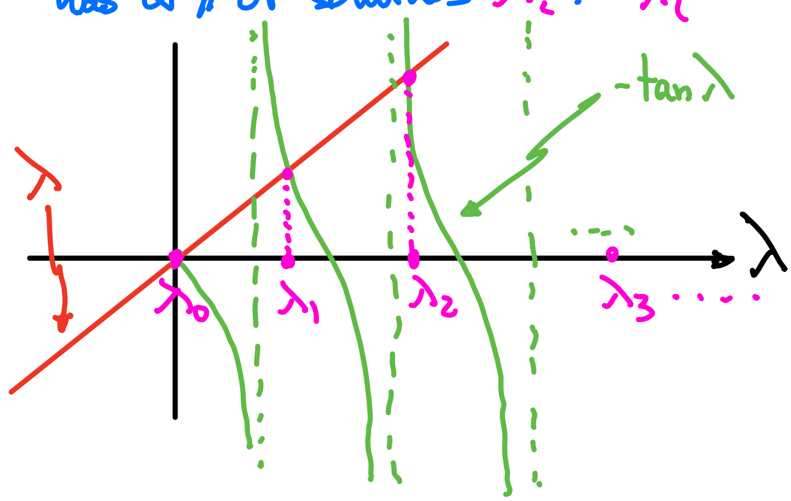
$y = c_1 \cos\lambda x + c_2 \sin\lambda x$

$y(0) = 0 = c_1$

$y(1) + y'(1) = c_2 \sin\lambda + c_2 \lambda \cos\lambda = 0$

$\therefore \; \lambda = -\tan\lambda \quad (\dagger)$

has ∞ # of solutions λ_i^*: $\lambda_i^* = -\tan\lambda_i^*$



For $\alpha = 0$

$y = c_1 x + c_2$

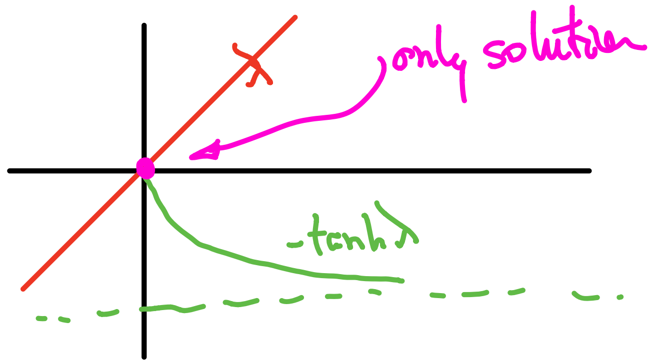
apply B.C. and get $y = 0$ as the only solution.

For $\alpha < 0$ here $\lambda = \sqrt{|\alpha|}$

$$y = c_1 \sinh \lambda x + c_2 \cosh \lambda x$$

Apply B.C. to get

$$\lambda = -\tanh \lambda$$



So the non-trivial solutions are

$$y = c\, \varphi_n(x, \lambda_n)$$

$n = 1, 2, \ldots$

where c is a constant

$\lambda_n > 0$ are roots of

$$\lambda_n = -\tan \lambda_n$$

$$\varphi_n = \sin \lambda_n x$$

Rmk: We will see that φ_n are eigenfunctions and λ_n are eigenvalues corresponding to the differential operator

$$-\frac{d^2}{dx^2}$$

with B.C.

$$\begin{cases} \varphi_n(0) = 0 \\ \varphi_n(1) + \varphi_n'(1) = 0 \end{cases}$$

STURM-LIOUVILLE BVP (SL)

Rule: SL BVP belong to a class of linear operators that are SELF ADJOINT

A bit of linear algebra:

Recall that a (square) matrix A is Hermitian if $A^* = A$

A^* is the conjugate transpose of A, and if A is real then $A^* = A^T$

Consider the eigenvalue problem

$$(\$) \quad Ax = \lambda x$$

where A is real and symmetric. A is then Hermitian. Hermitian matrices belong to a larger class of square matrices called normal, these are matrices that

$$A^* A = A A^* \quad \text{(commute), that are}$$

SELF ADJOINT.

The eigenvalues λ and eigenfunctions of A, satisfy ($) and are real. Moreover, the eigenfunctions form an orthogonal set. and are a basis for the column space of A: that is, if x_i and x_j are two eigenvectors of A with associated eigenvalues λ_i and λ_j then

$$x_i^T x_j = \delta_{ij} \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

This is orthogonality.

Hermitian matrices are self-adjoint.

The claim is that there are a set of linear differential operators L with boundary conditions that are self adjoint:

$$L y = \lambda r(x) y \,, \quad a < x < b$$

where

$$L y = \left\{ p_n(x) \frac{d^n}{dx^n} + p_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} \cdots + p_1(x) \frac{d}{dx} + p_0(x) \right\} y$$

$$r(x) > 0 \quad , \; a < x < b$$

n even

and n suitable B.C and certain conditions placed on $p_i(x)$.

We will touch upon 2nd order linear self-adjoint differential equations in what follows.

The point is that is the ODE+BVP is self adjoint, it too will be an eigenvalue problem with orthogonal eigenfunctions (possibly an infinite # of them) with real eigenvalues.
These eigenfunctions form a basis for the solution of the ODE + B.C.

The 2nd order Sturm Liouville Problem:

ODE $-[p(x)y']' + q(x)y = \lambda r(x)y, \quad 0<x<1$

B.C. $\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$

p, p', q, r continuous on $0 \leq x \leq 1$

$p(x) > 0$ and $r(x) > 0$ for $0 \leq x \leq 1$

let $Ly = -[p(x)y']' + q(x)y$

then ODE $Ly = \lambda r(x)y$

ex) $y'' + s^2 y = 0$ or $-y'' = s^2 y$

$y(0) = y(1) = 0$

here $\begin{cases} p = 1 \\ r = 1 \end{cases}$ and $\lambda = s^2$

both p & $r > 0$ over $0 \le x \le 1$. //

<u>Rule</u>: We said that SL BVP is self adjoint.

In the following calculation we see what consequences follow from this fact:

let $u(x), v(x) \in C^2[0,1]$ function that are continuous, with first and second derivatives continuous as well for $x \in [0,1]$.

let's compute

$$\int_0^1 (Lu)v\,dx = \int_0^1 \left(-(pu')'v + quv\right)dx$$

integrating by parts

$$\int_0^1 (Lu)v\,dx = -p(x)u'(x)v(x)\Big|_0^1 + p(x)u(x)v'(x)\Big|_0^1 + \int_0^1 \left[-u(pv')' + uqv\right]dx$$

$$= -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^1 + \int_0^1 u(Lv)\,dx$$

Grouping terms

$$\int_0^1 [(Lu)v - u(Lv)]\,dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]\Big|_0^1 \qquad (¥)$$

"Lagrange Identity"

S'pose $u(x)$ and $v(x)$ satisfy the B.C.:

$$\begin{cases} \alpha_1 \begin{pmatrix} u \\ v \end{pmatrix}(0) + \alpha_2 \begin{pmatrix} u \\ v \end{pmatrix}'(0) = 0 \\ \beta_1 \begin{pmatrix} u \\ v \end{pmatrix}(1) + \beta_2 \begin{pmatrix} u \\ v \end{pmatrix}'(1) = 0 \end{cases}$$

then (¥) leads to

$$\int_0^1 (Lu)v\,dx = \int_0^1 u(Lv)\,dx \qquad €$$

We will introduce traditional notation:

The inner product of 2 functions $u(x)$, $v(x)$ over a given interval $a \le x \le b$

$$(u,v) \equiv \int_a^b u(x)\bar{v}(x)\,dx = \int_a^b \bar{u}(x)v(x)\,dx$$

$(\bar{\ })$ is complex conjugate.

So $\mathcal{E}$ can be written as

$$(Lu,v)-(u,Lv)=0$$

<u>Thm</u>: All e'values of SL are real

<u>Pf</u>: let $u=v=\phi$, then

$$(L\phi,\phi)=(\phi,L\phi)$$

but $L\phi=\lambda r\phi$ $\therefore$

$$(\lambda\phi,\phi)=(\phi,\lambda r\phi)$$

or $\displaystyle\int_a^b \lambda r\phi\bar{\phi}\,dx=\int_a^b \phi\bar{\lambda}\bar{r}\bar{\phi}\,dx$

but $r(x)$ is real $\therefore$

$$(\lambda-\bar{\lambda})\int_a^b r\phi\bar{\phi}\,dx=0$$

if $\phi=\phi_r+i\phi_i$ (complex)

then $\phi\bar{\phi}=\phi_r^2+\phi_i^2$ which is real

$$\therefore\ (\lambda-\bar{\lambda})\int_a^b r(\phi_r^2+\phi_i^2)\,dx=0$$

since $\displaystyle\int_a^b r(\phi_r^2+\phi_i^2)\,dx$ cannot be 0

also $\displaystyle\int_a^b r(\phi_r^2+\phi_i^2)\,dx>0$

$\therefore \quad \lambda - \bar{\lambda} = 0 = (\lambda_r + i\lambda_i) - (\lambda_r - i\lambda_i)$

$= 2i\lambda_i = 0$

$\therefore \lambda_i$ must be 0

$\Longrightarrow \lambda$ must be real //

Thm: If ϕ_1 & ϕ_2 are any two e'functions of SL corresponding to λ_1 and λ_2 and $\lambda_1 \neq \lambda_2$ then

$$(\star) \qquad \int_a^b r(x)\phi_1(x)\bar{\phi}_2(x)\,dx = 0$$

That is, ϕ_1 and ϕ_2 are orthogonal over $x \in [a,b]$ wrt the WEIGHT $r(x)$!

Pf: Since $L\phi = \lambda r\phi$ & $(Lu,v) = (u,Lv)$

then $(\lambda_1 r\phi_1, \phi_2) - (\phi_1, \lambda_2 r\phi_2) = 0$

In HW you will show that ϕ, the e'functions of SL are REAL

$$\therefore (\lambda_1 - \lambda_2)\int_a^b r\phi_1\phi_2\,dx = 0$$

since $\lambda_1 \neq \lambda_2$ then $(\star)$ is shown //

Thm: The e'values of SL are SIMPLE. That is, to each e'value there corresponds ONLY ONE linearly independent eigenfunction. Further, the eigenvalues form an infinite sequence and can be ordered according to increasing magnitude

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

Moreover $\lambda_n \to \infty$ as $n \to \infty$

Pf: Sketch of proof assigned in HW. //

Returning to the connection between the eigenvalue problem for a Hermitian matrix A, that is

$$Ax = \lambda x \quad \text{where } A \in \mathbb{R}^{n \times n} \text{ (symmetric)}$$

Hermitian, there are n simple eigenvalues of A and n associated e'vectors of A. So the $Ax = \lambda x$ is a discrete counterpart of $L\phi = r\lambda\phi$. //

Rmk: It is often convenient to find eigenfunctions of a SL problem in NORMALIZED form. They are then called ORTHONORMAL

To find the normalization, we use

$$\int_a^b r(x)\,\phi^2(x)\,dx = N^2 \text{ (the normalization)}$$

then let $\hat{\phi}(x) = \dfrac{\phi}{N}$

$$\therefore \int_a^b r(x)\,\hat{\phi}^2(x)\,dx = 1$$

//

ex) Determine the normalized e'functions and eigenvalues of the following SL problem:

$$\begin{cases} y'' + \lambda^2 y = 0 & \text{ODE} \\ y(0) = y(1) = 0 & \text{B.C.} \end{cases}$$

1. First confirm that ODE + B.C. is an SL (we did that previously and found also that $r(x) = 1$.

$y = a\cos\lambda x + b\sin\lambda x$ solves ODE.

apply B.C.

$y(0) = a\cos(\lambda\cdot 0) + b\sin(\lambda 0) = a = 0$

$\therefore\ y = b \sin \lambda x$

$y(1) = 0 = b \sin \lambda$

since $b \neq 0$ then $\sin \lambda = 0$

which is true if $\lambda = n\pi$

$n = 1, 2 \ldots$

$\therefore\ \lambda_n = n\pi$

the eigenvalues are $\lambda_n^2 = n^2\pi^2$

The eigenfunctions are $\phi_n = \sin n\pi x$

$n = 1, 2 \ldots$

We can confirm that they are orthogonal:

$$\int_0^1 \sin n\pi x \sin mx \, dx = \begin{cases} \frac{1}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

so the orthonormal e'functions are

$$\hat{\phi}_n = \sqrt{2} \sin n\pi x$$

$n = 1, 2 \ldots$