We continue here with the discussion on stability of a (p+1) stage scheme

$$(\dagger) \quad y_{n+1} = y_n + \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j})$$

let $\rho(r) \equiv r^{p+1} - \sum_{j=0}^{p} a_j r^{p-j}$

Rmk: set $b_j = 0$, $\rho(r)$ is built upon the solution $y_n = r^p$ of the homogeneous difference equation.

$\rho(1) = 0$ since $\sum_{j=0}^{p} a_j = 1$

Set $\rho(r) = 0 \quad (\ast)$

and compute the roots of $(\ast)$:

$r_0, r_1, \ldots, r_p$.

We set $r_0 = 1$ (there must be a root $=1$).

We say that $(\ddagger)$ satisfies the root condition if

$$|r_j| \leq 1 \qquad j = 0, 1, \dots, p$$

if any of the $|r_j| = 1$ it must be a simple root.

<u>Thm</u> S'pse $(\ddagger)$ is consistent. Then $(\ddagger)$ is stable iff the root condition is satisfied.

<u>Pf</u> (not shown). //

ex) We saw previously that

$$y_{n+1} = 3y_n - 2y_{n-1} + \frac{h}{2}\left[f(x_n, y_n) - 3f(x_{n-1}, y_{n-1})\right]$$

is unstable.

This should be reflected in the root condition not being satisfied.

Compute $\rho(r) = r^2 - 3r + 2 = 0$

with roots $r_0 = 1$ and $r_1 = 2$

$\therefore$ root condition violated.

How does the analysis proceed for the system of equations?

$$\text{IVP}\begin{cases} Y' = f(x, Y) \\ Y(0) = Y_0 \end{cases} \qquad \begin{matrix} Y \in \mathbb{C}^n & Y_0 \in \mathbb{C}^n \\ f \in \mathbb{C}^n & \end{matrix}$$

We proceed (crudely) by linearizing.

let $Y(x) = \tilde{Y}(x) + \delta Y(x)$, assume that

$$\| \delta Y \| << 1$$

then $$\frac{d}{dx}\left(\tilde{Y}(x) + \delta Y(x)\right) = f\left(x, \tilde{Y}(x) + \delta Y(x)\right)$$

$$\tilde{Y}(0) + \delta Y(0) = Y_0 + \varepsilon$$

small $\|\varepsilon\|$

added perturbation to Y_0

If $\tilde{Y}(x)$ solves IVP

then

$$\begin{cases} \dfrac{d}{dx}\delta Y = \left.\dfrac{\partial f}{\partial Y}\right|_{Y=\tilde{Y}} \delta Y \\ \delta Y(0) = \varepsilon \end{cases}$$

is an equation for $\delta Y(x)$ dynamics.

$$\left.\frac{\partial f}{\partial Y}\right|_{Y=\tilde{Y}} = \left.\frac{\partial f_i}{\partial Y_j}\right|_{Y=\tilde{Y}}$$ is a matrix (the Jacobian)

(for $1 \le i, j \le n$)

$$\begin{cases} \dfrac{d}{dx}\delta Y = A(x)\,\delta Y \\ \delta Y(0) = \varepsilon \end{cases} \qquad A(x) = \left.\frac{\partial f}{\partial Y}\right|_{Y=\tilde{Y}}$$

In what follows, let's work with autonomous ODE

$$\frac{dY}{dx} = f(Y)$$

$\uparrow$ no x dependence

to make what follows more straightforward

Then the resulting perturbed system $W \equiv \delta Y(x)$ is, for $x \geq 0$

$$\begin{cases} \dfrac{dW}{dx} = AW, & A \in \mathbb{C}^{n \times n} \\ & W \in \mathbb{C}^n \\ W(0) = W_0 \end{cases}$$

Rmk: A quick aside, to remind you how

$$\frac{dw}{dx} = Aw, \quad w(0) = w_0, \quad x \geq 0.$$

let's assume that A has n distinct e'values $\{\lambda_i\}_{i=1}^{n}$ and e'vectors $\{v_i\}_{i=1}^{n}$ (other cases can be dealt with, but that's complication that obscures the presentation).

Guess that $w = e^{Ax} q$ is a solution to $w' = Aw$ (try it). Here e^{Ax} is a matrix exponential.

The e'values & e'vectors of A are found by solving

$$(A - I_n \lambda_i) v_i = 0 \qquad 1 \leq i \leq n$$

where $\lambda_i \in \mathbb{C}$

since $w = e^{Ax} e^{-\lambda I_n x} e^{\lambda I_n x} q$

then $w = e^{(A-\lambda I_n)x} e^{\lambda I_n x} q$

or $w = e^{\lambda I_n x} q$.

let $w_0 = \sum_{i=1}^{n} c_i v_i$ be an expansion of w_0 as a linear combination of e-vectors.

then $w(0) = w_0 = \sum_{i=1}^{n} c_i v_i$

$\therefore w(x) = \sum_{i=1}^{n} c_i v_i e^{\lambda_i x}$ //

So for $\frac{dw}{dx} = Aw = V^{-1}\Lambda V w$

$w(0) = w_0$

All we need to do is to study

$\frac{dv}{dx} = \Lambda v \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

$\lambda_i \in \mathbb{C}$

$V(0) = I_n$ ($n \times n$ identity //

We'll do so by focusing on

$$(\$) \begin{cases} \dfrac{dY}{dx} = \lambda Y & x > 0 \\ Y(0) = 1 \end{cases}$$

Let's get back to (‡). The multi-stage scheme will approximate solutions to (\$).
We use (\$) as a proxy for how the more complex IVP scheme responds to perturbations.

Rmk: The perturbations might be put in "by hand", but on a finite precision machine these perturbations can originate in the finite precision of the quantities.

Using (¢) on ($):

$$(¥) \quad y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h\lambda \sum_{j=-1}^{p} b_j y_{n-j}$$

or

$$(1 - h\lambda b_{-1}) y_{n+1} - \sum_{j=0}^{p} (a_j + h\lambda b_j) y_{n-j} = 0$$

for $n \geq p$.

Substitute $y_n = r^{n-p}$ and we get

the "characteristic equation"

$$\rho(r) - h\lambda \sigma(r) = 0$$

where $\sigma(r) \equiv b_{-1} r^{p+1} + \sum_{j=0}^{p} b_j r^{p-j}$

let $z \equiv h\lambda$

So $\rho(r) - z\,\sigma(r) = 0 \quad (*)$

let $r_0(z), r_1(z) \ldots r_p(z)$

be the roots of $(*)$ and they depend continuously on z. In particular

let $z \to 0$ then

$r_0(0), r_1(0) \ldots, r_p(0)$ are the roots

of $\rho(r) = 0$

with $r_0(0) = 1$.

Assuming the roots $r_i(z)$ are distinct, then

$$y_n = \sum_{j=0}^{p} \gamma_j \left[r_j(z) \right]^n \quad n \geq 0$$

is the solution to $(\not{y})$, γ_j are constants.

Rmk: when λ_i are not distinct we use the method of "reduction of order" (ODE's) to find the other solutions for the root(s) with multiplicity > 1.

For example, if $r_j(z)$ is repeated $\nu > 1$ times, the solutions are of the form

$$[r_j(z)]^n,\ n[r_j(z)]^n,\ n^2[r_j(z)]^n \cdots n^{\nu-1}[r_j(z)]^n .$$

Rmk: Recall that the root condition required $0 \le j \le p$: $|r_j| \le 1$, but if any r_i was had multiplicity > 1 then the root condition is $|r_i| < 1$ strictly.

Now you see why. These parts of the

solution y_n can grow algebraically

$\propto n, n^2, \ldots n^{\nu-1}$.