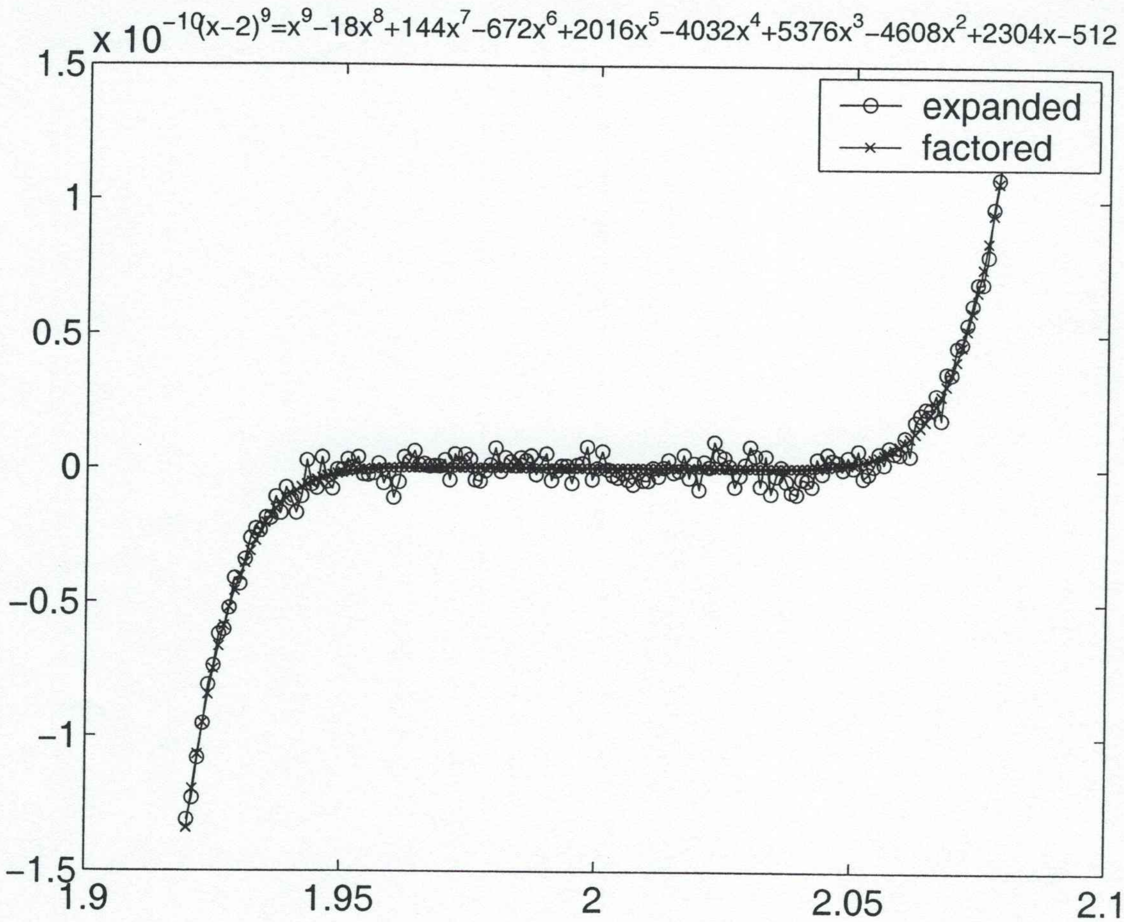
HW5 13.3



$(x-2)^9=x^9-18x^8+144x^7-672x^6+2016x^5-4032x^4+5376x^3-4608x^2+2304x-512$
x 10^{-10}
1.5
1
0.5
0
−0.5
−1
−1.5
1.9
1.95
2
2.05
2.1
expanded
factored

14.1 True or false?

(a) $\sin x = O(1)$ as $x \to \infty$

True $|\sin x| \leq 1$ $\forall x$

(b) $\sin x = O(1)$ as $x \to 0$

True, as above, but not sharp, as $|\sin x| \leq x$ $\forall x$
and so as $x \to 0$ $\sin x = O(x)$

(c) $\log x = O(x^{1/100})$ as $x \to \infty$

true, but not sharp: let $y = x^{1/100}$ so

$\log(x) = \log(y^{100}) = 100 \log y$.

For $y \geq e^7$, $|100 \log(y)| \leq y$

(check this: at $y = e^7$ compare the derivatives

$\frac{100}{y}$ and 1 for larger y's. Hence, for $x \geq e^{700}$

$\log(x) \geq x^{1/100}$

$$n! = O((n/e)^n) \text{ as } n \to \infty$$

(d) False. Stirling's formula gives an asymptotic formula for $n!$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\; n^n e^{-n}} = 1$$

so $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n e^{-n}} = \lim \frac{\sqrt{2\pi n}\; n^n e^{-n}}{n^n e^{-n}} = \lim_{n\to\infty} \sqrt{2\pi n} = \infty$

(e) True, $A = O(V^{2/3})$ as $V \to \infty$

where A sphere surface, V its volume.

$V = \frac{4}{3}\pi r^3$ so $V^{2/3} = \left(\frac{4\pi}{3}\right)^{2/3} r^2 = \alpha r^2$

and $A = 4\pi r^2 = \beta r^2$

$\displaystyle\lim_{V\to\infty} \left(\frac{A}{V}\right) = \frac{\beta}{\alpha}$, constant, units are irrelevant.

f) $fl(\pi) - \pi = O(\varepsilon_{mach})$ true as a consequence of (13.5) where the constant is π.

(g) $fl(n\pi) - n\pi = O(\varepsilon_{mach})$ uniformly for n integer.

False: (13.5) tells us that $fl(n\pi) - n\pi = \varepsilon n\pi$ for <u>some</u> ε with $|\varepsilon| \le \varepsilon_{mach}$. That means that the constant should be proportional to n

15.1 Using a computer that satisfies axioms (13.5) and (13.7) we perform the following operations. For each state whether algorithm is backward stable, stable but not backward stable, or unstable.

(a) Data $x \in \mathbb{C}$. Solution: $2x$ computed as $x \oplus x$

is backward stable. Using (13.5) $fl(x) = x(1+\varepsilon_1)$ $|\varepsilon_1| \leq \varepsilon_m$.

(13.7) gives $fl(x) \oplus fl(x) = 2fl(x)(1+\varepsilon_2)$

$|\varepsilon_2| \leq \varepsilon_m$.

$\therefore$ $fl(x) \oplus fl(x) = 2x(1+\varepsilon_1)(1+\varepsilon_2)$. Let

$\tilde{x} = x^2(1+\varepsilon_1)(1+\varepsilon_2)$ $\Rightarrow$ we have

$fl(x) \oplus fl(x) = fl(\tilde{x})$ and $\tilde{x} - x = x(\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2)$

that is,

$$\frac{|\tilde{x} - x|}{|x|} = O(\varepsilon_m).$$

(b) Data $x \in \mathbb{C}$. Solution x^2 computed $x \otimes x$
Backward stable.
Using (13.5) $fl(x) = x(1+\varepsilon_1)$ $|\varepsilon_1| \le \varepsilon_m$
(13.7) gives $fl(x) \otimes fl(x) = x^2(1+\varepsilon_1)^2(1+\varepsilon_2)$

Choose $\tilde{x} = x(1+\varepsilon_1)\sqrt{1+\varepsilon_2}$ $\Rightarrow$

$fl(x) \otimes fl(x) = \tilde{x}^2$ and since $\sqrt{1+O(\varepsilon_m)} = 1+O(\varepsilon_m)$

and $(1+O(\varepsilon_m))(1+O(\varepsilon_m)) = 1+O(\varepsilon_m)$ we have backward stability.

(c) Data $x \in \mathbb{C}\setminus\{0\}$. Solution: 1, computed via $x \oslash x$

stable, but not backward stable. Using 13.5

$fl(x) = x(1+\varepsilon_1)$ and 13.7 $fl(x) \oslash fl(x) = 1(1+\varepsilon_2)$

There's no input to the exact problem that produces

any answer other than 1. Hence algorithm cannot

be backward stable. However the ratio of the error in the result to the correct result is $O(\varepsilon)$, so

algorithm is stable.

15.2 Consider an algorithm for the problem of computing the full SVD of a matrix. The data for this problem is a matrix A and the solution is U, Σ, V such that $A = U\Sigma V^*$
(We are speaking of explicit matrices U & V, not implicit representations as products of reflectors)

(a) Explain what it would mean if this algorithm was backward stable.
(b) In fact, for a simple reason, it cannot be backward stable. Explain
(c) Fortunately, the standard algorithm for computing SVD (lecture 31) are stable. Explain what stability means for such algorithm.

Input $A \in \mathbb{C}^{m\times n}$ and output $\hat{U}, \hat{V}, \hat{\Sigma}$ approximating U, V, Σ, with U $m\times m$ unitary, V $n\times n$ unitary and Σ diagonal $m\times n$ matrix.

(a) For algorithm to be backward stable, there would be $\hat{A} = A + \delta A$ with

$$\hat{A} = \hat{U}\hat{\Sigma}\hat{V}^* \quad \text{and} \quad \frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m),$$

(b) The dimension of the space into which the results are calculated is $n^2 + m^2 + n$ (assuming $n \le m$ and that the diagonal is not fully calculated, just the vector of diagonal matrices). The dimension of the source is nm which is smaller ∴ we should not expect BW stability.