9.2 In Experiment 2, the singular values of A match the diagonal elements of a QR factor R approximately. Consider now a very different example. Suppose $Q = I$ and $A = R$, the $m \times m$ Toeplitz matrix with 1 on the main diagonal, 2 on the first superdiagonal, and 0 everywhere else.

(a) What are the eigenvalues, determinant and rank (A)?

Since A is upper diagonal, its eigenvalues are its diagonal entries which are all 1. Its determinant equal to the product of the eigenvalues and thus 1. The rank of this nonsingular matrix is m.

(b) What is A^{-1}?

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & -8 & \cdots & -2^{m-1} \\ 0 & 1 & -2 & 4 & \cdots & -2^{m-2} \\ 0 & 0 & 1 & -2 & \cdots & -2^{m-3} \\ 0 & 0 & 0 & 1 & \cdots & -2^{m-4} \\ \vdots & \vdots & \vdots & 0 & & \vdots \\ 0 & 0 & 0 & 0 & \cdots 0 & 1 \end{pmatrix} \quad \text{or} \quad a_{ij} = \begin{cases} 0 & \text{if } i > j \\ \vdots \\ -2^{j-1} & \text{otherwise} \end{cases}$$

(c) let $A = U\Sigma V^*$ then $A = V\Sigma^{-1}U^*$

and the singular values of A^{-1} are inverses of those of A

Hence $1/\sigma_m$ is the largest singular value and hence $1/\sigma_m = \|\Delta^{-1}\|_2$. We need an upper bound on σ_m which translates into a lower bound on $\|\Delta^{-1}\|_2 = \max\limits_{\|v\|_2=1} \|\Delta^{-1}v\|_2$

So in particular if we take $v = e_m$ then $\Delta^{-1}e_m$ is the m^{th} column of Δ^{-1} whose norm is $\sqrt{\sum\limits_{j=0}^{m-1} 2^{2j}} = \sqrt{2^m-1}$ $\therefore$

$$\frac{1}{\sigma_m} \geq \sqrt{(4^m-1)/3}.$$

(it turns out to be a not so sharp estimate).