5.3 (T.B.) $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$

(a) ~~Determine~~ SVD $A = U\Sigma V^T$. The decomposition is not unique. Find the one that has minimal number of minus signs in U & V.

The eigenvalues of A are straightforward to obtain and thus $\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$

I'll pick $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so by inspection $V = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

(c) Find the 1-, 2-, ∞- & Frobenius norms:

$\|A\|_1 = \max_i \|a_i\|_1$ $\|A\|_2 = 10\sqrt{2}$

$\|A\|_\infty = \max_i \|a_i^*\|_1 = 15$ $\|A\|_F \approx 15.8 = 5\sqrt{10}$

(d) $A^{-1} = V\Sigma^{-1}U^T = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.2 \end{bmatrix}$

(e) $\lambda_{1,2} = 1.5 \pm i\,\frac{1}{2}\sqrt{391}$

(f) $\det(A) = \sigma_1\sigma_2 = 100$

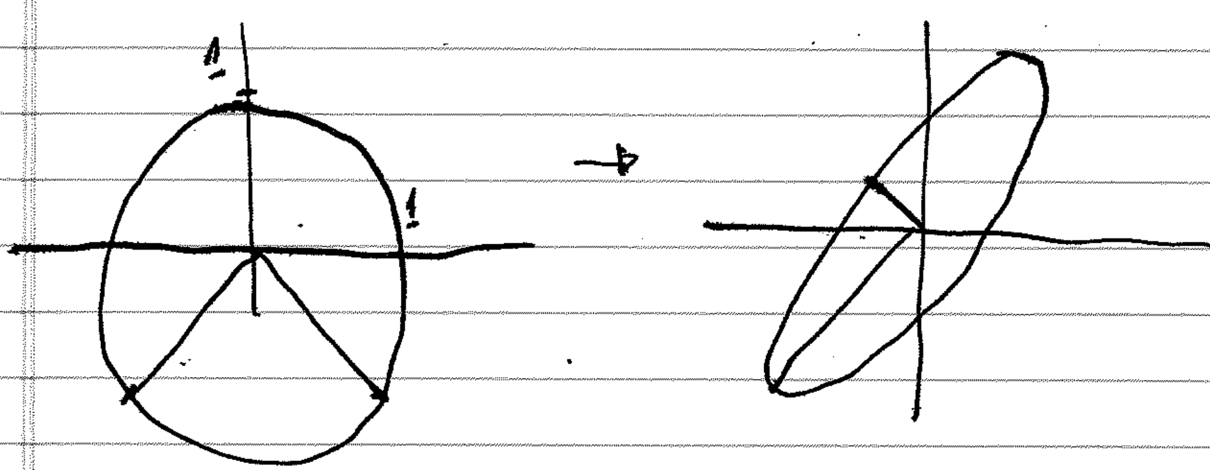
(g) Area of ellipse $= ab\pi = 100\pi$. The matrix A maps the unit $L_2$ ball in $\mathbb{R}^2$ to ellipse with axes $\sigma_1$ & $\sigma_2$.

(b) The right singular vectors

$$\pm \frac{1}{5}\begin{bmatrix} -3 \\ 5 \end{bmatrix} \text{ and } \pm \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The left singular vectors

$$\pm \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \pm \begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}}$$



5.4(TB) Suppose $A \in \mathbb{C}^{m \times m}$, $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $B \in \mathbb{C}^{2m \times 2m}$ Hermitian

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

let $z = [x \quad y]^T$ $\quad x, y \in \mathbb{C}^m$

the eigenvalue problem $Bz = \lambda z$ is

$$\begin{cases} Ax = \lambda y \\ A^* y = \lambda x \end{cases} \quad (1)$$

let $U = [u_1 u_2 \cdots u_m]$ and $V = [v_1 v_2 \cdots v_m]$

$v_i$ & $u_i$ are the $i^{th}$ column vectors of $V$ & $U$.

know that $\begin{cases} Av_i = \sigma_i u_i \\ A^* u_i = \sigma_i v_i \end{cases} \quad (2)$

Note: $u_i$ & $v_i$ are orthogonal.

Comparing (1) & (2) we see $z_i = \begin{bmatrix} v_i \\ u_i \end{bmatrix}$ is an eigenvector

of $B$ with eigenvalue $\sigma_i$

The goal is to find an orthogonalization of $B = P \Lambda P^*$

We know the (matrix) entry $\Lambda_i = \begin{bmatrix} \sigma_i & \phi \\ \phi & ? \end{bmatrix}$. We guess $-\sigma_i$, so more properly

$$\Lambda = \begin{bmatrix} \Sigma & \phi \\ \phi & -\Sigma \end{bmatrix}$$

the e'vector associated with $-\sigma_i$ is $\begin{bmatrix} v_i \\ -u_i \end{bmatrix}$

We want $P$ to be orthonormal so

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \qquad P^* = \frac{1}{\sqrt{2}} \begin{bmatrix} V^* & U^* \\ V^* & -U^* \end{bmatrix}$$

$$\therefore B = \frac{1}{2} \begin{bmatrix} V & V \\ U & -U \end{bmatrix} \begin{bmatrix} \Sigma & \phi \\ \phi & -\Sigma \end{bmatrix} \begin{bmatrix} V^* & U^* \\ V^* & -U^* \end{bmatrix}$$

TB 6.1 If $P$ is an orthogonal projection, then $I-2P$ is unitary.
Prove & show geometric interpretation:

$$(I-2P)^*(I-2P) = (I-2P)(I-2P) = I - 4P + 4P^2$$
but $P^2 = P$ so $\quad = I - 4P + 4P = I$.

TB 6.3 Given $A \in \mathbb{C}^{m\times n}$ $m \geq n$ show that $A^*A$ is nonsingular iff $A$ has full rank

If $A^*A$ is singular $\lambda = 0$ is an e'value of $A^*A$. Only if $A^*A$ is nonsingular $\lambda = 0$ is NOT an e'value & hence no singular values equal to 0.

A zero singular value would make it possible for $A \in \mathbb{C}^{m\times n}$ $m \geq n$ to map two distinct vectors to the same vector (rank deficient);

Geometrically: let $A = U\Sigma V^*$. $A$ will have full rank if $\Sigma$ has nonzero diagonal elements.
Since $A^*A = V\Sigma^*\Sigma V^*$ we see that $A^*A$ is nonsingular if $\Sigma$ has maximal rank $\therefore$ $A^*A$ is nonsingular iff $A$ has full rank.

TB 6.4 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(a) What is the $\perp$ projector $P$ onto range $(A)$ and what is the image under $P$ of the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$?

$$P = A(A^*A)^{-1}A^* = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \qquad P\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

(b) Same for $B$:

$$P = B(B^*B)^{-1}B^* = \frac{1}{6}\begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{pmatrix} \qquad P\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

TB 7.1 Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Using any method find $A = \hat{Q}\hat{R}$ (reduced) and $A = QR$ (full) factorizations

(b) Same for $B$

(a) The columns of A are already orthogonal

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Normalizing the columns

(b) Use Gram Schmidt (Alg 7.1)

$$B = (b_1\ b_2)$$

$$(b_1, b_2) = (q_1\ q_2)\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

$$r_{11} = \|b_1\|_2 = \sqrt{2} \qquad q_1 = \frac{b_1}{\|b_1\|_2} = (1/\sqrt{2}, 0, 1/\sqrt{2})^T$$

$$r_{12} = q_1^* b_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}$$

$$b_2 - r_{12} q_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{2}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$r_{22} = \sqrt{3}$$

$$B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

7.5 Let $A$ be $m \times n$ $(m \geq n)$ matrix. Let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that $A$ has rank $n$ iff all diagonal elements of $\hat{R}$ are nonzero

(b) Suppose $\hat{R}$ has $k$ nonzero diagonal entries for some $k$ with ~~[illegible]~~ $0<k<n$. What does this imply about rank of $A$? Exactly $k$? At least $k$? At most $k$?

(a) Examine nullspace problem

$$Ax = 0$$
$$\hat{Q}\hat{R}x = 0$$
$$\hat{R}x = 0$$

$\hat{R}$ is upper triangular. If not all diagonal elements of $\hat{R}$ are nonzero $\Rightarrow \det(\hat{R}) = 0$, which implies that $\hat{R}$ is singular and the nullspace of $\hat{R}$ is not empty.

Hence $\hat{R}x = 0$ for some $x \neq 0$ and then $Ax = 0$ for the same $x \neq 0$ which implies that $A$ is rank deficient $\therefore$ only if all elements of the diagonal of $\hat{R}$ are nonzero, $A$ has full rank.

(b) Some intuition first: The rank of A can never be less than k because, regardless, any off-diagonal elements in $\hat{R}$

$$\begin{bmatrix} q_1 \, q_2 \cdots q_k \cdots q_n \end{bmatrix} \begin{bmatrix} r_1 & & & & & & \\ & r_2 & & & & & \\ & & \ddots & & & & \\ & & & r_k & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

There are never less than k orthogonal vectors $q_i r_i$ and the rank of A is either exactly k or bigger than k.

See 2x2 example:

$$k=0 \qquad \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 0 & r_{12} \\ 0 & 0 \end{bmatrix} = r_{12} \begin{bmatrix} 0 & q_1 \end{bmatrix} = A$$

$$\text{rank}(A) = 1 > k$$

$\therefore$ rank(A) $\geq k$ (the rank of A is at least k).

See a rank(A) $= 2 > k = 1$ example:

$$A = \hat{Q}\hat{R} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = (q_1, 0, q_2)$$

//

8.2 Download eighttwo.m, write a code ~~[Q,R]=...~~

$$[Q,R] = \text{mgs}(A)$$

that computes a reduced QR factorization $A = \hat{Q}\hat{R}$ of an $m \times n$ matrix with $m \geq n$ using modified Gram-Schmidt Orthogonalization (The output should be $Q \in \mathbb{C}^{m \times n}$ with orthogonal columns & a triangular matrix $R \in \mathbb{C}^{n \times n}$. Turn in the output for eighttwo for $n = 80 = m$; $n = 80, m = 160$; $n = 80, m = 320$ and explain what the resulting output means.

SEE CODES

should demonstrate that the factorization recovers A, to numerical precision and that the columns of Q are orthogonal, to numerical precision.

8.3 The upper triangular matrix $R_i$ can be rewritten as a product of two matrices:

$$R_i = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \frac{1}{r_{i,i}} & \frac{-r_{i,i+1}}{r_{i,i}} & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \frac{1}{r_{i,i}} & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}}_{R_i^{(1)}} \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & -r_{i,i+1} \\ & & 1 \end{bmatrix}}_{R_i^{(2)}}$$

The product with $R_i^{(1)}$ (diagonal) is the normalization corresponding to the line ~~$q_i = \frac{v_i}{r_{i,i}}$~~

$$q_i = \frac{v_i}{r_{i,i}} \quad \text{in Algorithm 8.1}$$

and the product with $R_i^{(2)}$ (the unit upper triangular) corresponds to the subtraction of $r_{i,j}$ times the resulting column $q_i$ from the remaining columns. In the algorithm this is represented by $v_j = v_j - r_{ij} q_i$