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TB 1.1 Let B be a 4x4 matrix to which the following operations are applied:

1) double column 1

$$B\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \equiv BA_1$$

2) halve row 3

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{bmatrix}BA_1 \equiv C_1BA_1$$

3) add row 3 to row 1

$$\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}C_1BA_1 \equiv C_2C_1BA_1$$

4) interchange columns 1 and 4

$$C_2C_1BA_1\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix} \equiv C_2C_1BA_1A_2$$

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5) subtract row 2 from each of the other rows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} C_2 C_1 B \Delta_1 \Delta_2 \equiv C_3 C_2 C_1 B \Delta_1 \Delta_2$$

6) Replace column 4 by column 3:

$$C_3 C_2 C_1 B \Delta_1 \Delta_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \equiv C_3 C_2 C_1 B \Delta_1 \Delta_2 \Delta_3$$

7) Delete Column 1

$$C_3 C_2 C_1 B \Delta_1 \Delta_2 \Delta_3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \underbrace{C_3 C_2 C_1}_{C} \underset{B}{B} \underbrace{\Delta_1 \Delta_2 \Delta_3 \Delta_4}_{A}$$

$$C = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} ; \Delta = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The key is to realize that <u>row</u> ops are performed by LEFT multiplies and <u>column</u> ops are performed by RIGHT multiplies.

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TB 1.1 The key is to realize that row operations are effected by LEFT multiplies and column operations are effected by RIGHT multiplies.

(a)
$$\underbrace{\underset{5.8.}{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}} \underset{3.}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{2.}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}} B \underbrace{\underset{1.}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{4.}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{6.7.}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

(b)
$$A = \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad B \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

T.B. 1.3 Generalizing example 1.3 consider a square or rectangular upper-triangular matrix ($r_{ij} = 0$ for $i > j$). By considering what space is spanned by the first n columns of R & using (1.8) show that if R is nonsingular $m \times n$ upper triangular matrix $\Rightarrow R^{-1}$ is also upper triangular (The analogous holds for low triangular matrices).

Let $I = SR$, $S = R^{-1}$. From Theorem 1.3, since R is invertible its determinant is nonzero

$$\det(R) = \prod_{i=1}^{m} r_{ii} \qquad \text{so } r_{ii} \neq 0 \text{ (must be!)}$$

$I = SR$ is written as $e_j = \sum_{k=1}^{m} r_{kj} s_k$

but since $r_{ij} = 0$ for $i > j$ then

$$e_j = \sum_{k=1}^{j} r_{kj} s_k$$

so $$s_k = \frac{1}{r_{jj}} e_j - \frac{1}{r_{jj}} \sum_{k=1}^{m-1} r_{kj} s_k$$

If we assume $s_{pq} = 0$ for $q < j$ & $p > q$ (i.e. that the first $j-1$ columns are "upper triangular") then it follows that $s_{pj} = 0$ for $p > j$ (since e_j has that property as well).

This sets up an induction to show that S is upper triangular. Once we show this is true for a single starting case. So, consider $j = 1 \Rightarrow s_1 = \frac{1}{r_{11}} e_1$, so $s_{p1} = 0$ for $p > 1$

Now, since we have a starting case ($j = 1$) and an induction step, we can conclude that $s_{ij} = 0$ for $i > j$ (S is upper triangular)

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TB 1.4 let $f_1, f_2 .. f_8$ be a set of functions defined on the interval [1, 8] with the property that for any numbers $d_1, d_2 .. d_8$ $\exists$ a set of coefficients $c_1, c_2, .., c_8$ st

$$\cancel{\sum c_j f_j(8)}$$

$$\sum_{j=1}^{8} c_j f_j(i) = d_i \qquad i = 1, 2, .., 8$$

(a) show that d_i's determine c_i's uniquely by Theorem 1.3.

let $g_j = [f_j(1)\ f_j(2) \dots f_j(8)]^T \qquad 1 \le j \le 8$

so the question is whether $\{g_j\}_{j=1}^{8}$ span $\mathbb{C}^8$

let $G = \begin{bmatrix} g_1 & g_2 & \dots & g_8 \end{bmatrix}$ then by Theorem 1.2 G is 1 to 1

Restating this in terms of c_i & d_i: for a given d $\exists$ a unique $C = G^{-1} d$

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(5) Let A be 8×8 matrix representing the linear mapping from data $d_1, d_2 .., d_8$ to coefficients $c_1, c_2 .., c_8$. What is the (i,j) entry of A^{-1}?

$\cancel{Ad_{i,j}}$ $\quad Ad = c \qquad c = G^{-1}d$

$d = A^{-1}c$

$\therefore AA^{-1}c = c \qquad \therefore \boxed{G = A^{-1}}$

T.B 2.1 Show that if a matrix A is both triangular & unitary then it is diagonal.

By problem 1.3 we had that if A is triangular, so will A^{-1} be. If unitary $A^* = A^{-1}$. By problem 1.3, in fact if A is upper (lower) triangular then A^{-1} will be upper (lower) triangular.

We must have $a_{ij} = 0$ if $i \neq j$:

We must have $a_{ii} a^*_{ii} = 1$ and $a_{ii} a^{-1}_{ii} = 1$ $1 \leq i \leq n$

The off diagonal elements of $A^{-1}A = \mathbb{I}$ are 0.

Since $a_{ii} \neq 0$ and these show up in the off diagonal elements of $A^{-1}A$ then a_{ij} for $i \neq j$ must be zero.

Hence A triangular & unitary must also be diagonal.

TB 2.3 Let $A \in \mathbb{C}^{m \times m}$ hermitian. An eigenvector of A is $Ax = \lambda x$, $x \neq \emptyset$, $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^m$. λ is the eigenvalue

(a) Prove all e'values of A are real:

$Ax = \lambda x$ $\quad A^* \bar{x} = \bar{\lambda} \bar{x}$ hence

$\cancel{(Ax)^* \bar{x} = (\lambda x) A^* \bar{x} = \bar{\lambda} x A \bar{x}}$

$$\bar{x} A x = \bar{x} \lambda x \Rightarrow \lambda = \frac{\bar{x} A x}{\bar{x} x} = \frac{(Ax)^* x}{\bar{x} x} = \frac{\bar{\lambda} \bar{x} x}{\bar{x} x}$$

$\cancel{\bar{\lambda} = \frac{\overline{\bar{x} A x}}{\bar{x} x} = \frac{x A^* \bar{x}}{x \bar{x}}}$

$\therefore \lambda = \bar{\lambda}$ $\therefore \lambda$ real.

Note: we used the fact that $\bar{x} x \neq 0$.

(b) Prove that if x & y are eigenvectors corresponding to distinct e'values $\Rightarrow$ x & y are orthogonal:

Let $u \to \lambda$ $\quad$ $\lambda \neq \mu$ are eigenvalues

$v \to \mu$

u, v are e'vectors

$$(\lambda - \mu) u^* v = \lambda u^* v - \mu u^* v$$

$$= (\bar{\lambda} u)^* v - u^* \mu v$$

$$= (Au)^* v - u^* A v = u^* A^* v - u^* A v = u^* A v - u^* A v = 0$$

$\therefore (\lambda - \mu) u^* v = 0$, since $\lambda \neq \mu \Rightarrow u^* v = 0$, orthogonal

T.B. 2.4 What can be said about the e'values of a unitary matrix?

$Q^{-1} = Q^*$ w/ (λ, x) e'value/vector pair

$$Qx = \lambda x$$

$$\therefore \|Qx\| = \sqrt{(Qx)^*(Qx)} = \sqrt{x^* Q^* Q x} = \sqrt{x^* x} = \|x\|$$

$$\therefore \|Qx\| = \|x\| = \|\lambda x\| = |\lambda| \, \|x\|$$

$\therefore |\lambda| = 1$ all e'values of Q lie on the unit complex circle and represent pure rotations.

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2.6 If u & v are m-vectors, the matrix $A = I + uv^*$ is known as the rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for α scalar. Find α. For what u, v is A singular? If it is singular what is null(A)?

$$I = AA^{-1} = (I + uv^*)(I + \alpha uv^*) = I + (1+\alpha)uv^* + \alpha uv^* uv^*$$

$$\therefore \quad (1+\alpha)uv^* = -\alpha uv^* uv^* \quad \text{or} \quad (1+\alpha) = -\alpha uv^*$$

$$\text{so } 1 = -\alpha - \alpha uv^* = -\alpha(1 + uv^*)$$

$$\therefore \alpha = -\frac{1}{1+uv^*} \quad \text{as long as } uv^* \neq 1$$

$$A^{-1} = I - \frac{uv^*}{1+uv^*}$$

$$\text{If } uv^* = -1 \Rightarrow (I + uv^*)u = u + uv^*u = u - u = 0$$

in which case $A = I + uv^*$ is not invertible since it has a non-zero null space.

The null space $Ax = 0 \Rightarrow x = -(v^*x)u$

$\therefore x$ is parallel to u (or v^*) Null(A) = span(u)

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~~TB 2.6 If u & v are m-vectors~~

T.B. 3.2 Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \le \|A\|$ where $\rho(A)$ is the "spectral radius" of A (i.e. the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

For any x $\quad \|Ax\| \le \|A\|\,\|x\|$

a property of any induced norm.

Suppose $\lambda x = Ax$

$$|\lambda|\,\|x\| = \|\lambda x\| = \|Ax\| \le \|A\|\,\|x\|$$

$\therefore\ |\lambda| \le \|A\|$ $\quad$ let $\rho(A) = \max\limits_{\lambda} |\lambda|$

hence $\rho(A) \le \|A\|$

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TB 3.3 Verify the following inequalities and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. $x \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m \times n}$

(a) $\|x\|_\infty = \max_i |x_i| = \max_i \left(|x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \|x\|_2$

$\therefore \|x\|_\infty \leq \|x\|_2$ equality is achieved for x with a single nonzero entry.

(b) $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \leq \left(\sum \max_i |x_i|^2\right)^{1/2} = \left(m \|x\|_\infty^2\right)^{1/2}$

$\therefore \|x\|_2 \leq \sqrt{m}\, \|x\|_\infty$

Take $x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ then $\|x\|_2 = \sqrt{m}$

and $\|x\|_\infty = 1$

(c) $\|A\|_\infty = \sup_{x \neq 0} \dfrac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sup_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_\infty} \leq \sup_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2 / \sqrt{n}}$

$= \sqrt{n}\, \|A\|_2$

For equality $A \in \mathbb{C}^{m \times n} = \left.\begin{pmatrix} \overbrace{1\,1\,1 \cdots 1}^{n} \\ 0 \end{pmatrix}\right\} m$

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$$A^*A = \begin{array}{c} \overbrace{\begin{pmatrix} \text{matrix} \\ \text{all ones} \\ \end{pmatrix}}^{n} \end{array} \Big\} n$$

has rank 1, with eigenvalues n and 0

0 is repeated e'value of multiplicity $n-1$.

So $\|A\|_2 = \sqrt{\rho(A^*A)} = \sqrt{n}$

$\|A\|_\infty = n \qquad \therefore \quad \|A\|_\infty = \sqrt{n}\,\|A\|_2$

(d) $\displaystyle \|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x\neq 0} \frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_2} \leq \frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_\infty}$

$\therefore \|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$

For equality $A \in \mathbb{C}^{m\times n} = \begin{array}{c} \overbrace{\begin{pmatrix} 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix}}^{n} \end{array} \Big\} m$

$$A^*A = \begin{array}{c} \overbrace{\begin{pmatrix} m & 0 & \cdots \\ 0 & & \\ \vdots & 0 & \end{pmatrix}}^{n} \end{array} \Big\} n$$

one nonzero entry equal to m.

$\therefore \|A\|_2 = \sqrt{\rho(A^*A)} = \sqrt{m}$

$\|A\|_\infty = 1$

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TB 4.1 Determine the SVD's by hand of

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = A = U\Sigma V^*$

Generally $\begin{cases} A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = VDV^* & \text{where } D = \Sigma^*\Sigma \\ AV = U\Sigma \end{cases}$

for A above we know $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

by inspection $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

so $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} V^*$ so $V^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = U\Sigma V^*$ U is 3×3 Σ is 3×2 V^* is 2×2

$A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = V\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}V^*$, by inspection $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$D = \Sigma^*\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$ by inspection $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\cancel{U\Sigma = AV\Sigma^-}$ $AV = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = U\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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(e) ~~A^*A~~

We can diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = ZDZ^*$

Computing the e'values & e'vectors $\lambda_1 = 2$, $\lambda_2 = 0$

$$z_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad z_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$Z = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = ZDZ^* = U\Sigma V^* = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

4.4 Two matrices A & $B \in \mathbb{C}^{m \times m}$ are "unitarily equivalent" if

$$A = QBQ^*, \quad Q \in \mathbb{C}^{m \times m} \text{ unitary.}$$

Is it true that A & B are unitarily equivalent iff they have the same singular values?

$$Ax = \lambda x$$

$$QBQ^* x = \lambda x \Rightarrow \underbrace{Q^*Q}_{I}BQ^* x = Q^* \lambda x$$

$B(Q^* x) = \lambda(Q^* x)$ which implies that A & B have same eigenvalues.