HW 10

24.1 For each statement prove it is true or give example if false. Assume $A \in \mathbb{C}^{m \times m}$ unless otherwise indicated. "ew" stands for eigenwert (eigenvalue).

(a) If λ is an ew of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an ew of $A - \mu I$

True. let v evector of A $\Rightarrow Av = \lambda v$
Also $\mu v = I \mu v$ $\therefore$ Adding both

$$(A - \mu I) v = (\lambda - \mu) v$$

(b) If A is real and λ an ew of A, then so is $-\lambda$:
False. See (a), pick $\mu = -\lambda$. Also easy to see for $A = I$

(c) If A is real and λ an ew of A, then so if $\bar{\lambda}$:
True.
let $p(A) = p(\lambda)$ be the characteristic polynomial of A:

$$p(\lambda) = \lambda^m + \alpha_{m-1} \lambda^{m-1} + \dots \alpha_1 \lambda + \alpha_0 = 0$$

if A is real $\Rightarrow \{\alpha_i\}_{i=0}^{m-1}$ are real

$$\overline{p(\lambda)} = \bar{\lambda}^m + \alpha_{m-1} \bar{\lambda}^{m-1} + \dots \alpha_1 \bar{\lambda} + \alpha_0 = 0$$

since d_i are real. So same polynomial $\therefore$ same roots.

(d) If λ is an ew of A & A is nonsingular then λ^{-1} is an ew of A^{-1}

True. $Av = \lambda v$, multiply by A^{-1} and divide by λ.

(e) If all ew's of A are zero, then $A = 0$.
False. Take $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ has all e'ws $= 0$.

(f) If A is Hermitian and λ is an ew, then $|\lambda|$ is a singular value of A

True. Diagonalizing: $A = Q \Lambda Q^*$ has real ews.
SVD $A = U \Sigma V^*$
so, upto the sign the singular values and the ew are the same $\therefore$ $\sigma_i = |\lambda_i|$

(g) If A is diagonalizable and all its ew's are equal then A is diagonal.

True $A = X \Lambda X^{-1}$ with $\Lambda = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \ddots \; \lambda \end{bmatrix} = I\lambda$

so $X^{-1}A = \Lambda X^{-1} \Rightarrow A = \Lambda X^{-1} X = \lambda X^{-1} X = \lambda I$ //

38.5 CG is an iterative minimization of the function $\varphi(x)$ is (38.7). Another way to minimize the same function - far slower in general - is by the method of STEEPEST DESCENT:

(a) Derive the formula $\nabla\varphi(x) = -r$ for the gradient of $\varphi(x)$. Thus the steepest descent iteration corresponds to the choice $p_n = r_n$ instead of $p_n = r_n + \beta_n p_{n-1}$ in Algorithm (38.1).

$$\varphi(x) = \frac{1}{2}x^T A x - b^T x$$

$$\nabla\varphi = Ax - b = -r$$

(b) Determine the formula for the optimal length α_n of the steepest descent

since $x_n = x_{n-1} + \alpha_n r_{n-1}$, the optimal α_n

minimizes $\varphi(x_n)$. So $\nabla_{\alpha_n}\varphi(x_n) = 0$

yields optimal α_n:

$$r_{n-1}^T A r_{n-1} \alpha_n + r_{n-1}^T (A x_{n-1} - b) = 0$$

solving for $\alpha_n = \dfrac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}}$