LECTURE 28 QR w/o Shifts

① Key observation: QR is a stable procedure for computing factorizations of matrix powers: $A, A^2, A^3, \ldots$

② Under suitable assumptions the iteration

$$A^{(0)} = A$$

$$\begin{array}{l} \text{for } k = 1, 2, \ldots \\ \quad Q^k R^k = A^{k-1} \\ \quad A^k = R^k Q^k \end{array}$$

will converge, as $k \to \infty$ to a Schur form (or to diagonal form if A is Hermitian)

Note: $A^k = (Q^k)^T A^{k-1} Q^k$

$\therefore A^k = (Q^0 Q^1 \cdots Q^k)^T A (Q^0 Q^1 \cdots Q^k)$

Whether Schur or diagonal

the $\operatorname{diag}(A^k) \approx \Lambda$,

As we did previously,

assume $v_0 = \sum_{i=1}^{m} a_i q_i$

$$A^k v_0 = \lambda_1^k a_1 q_1 + \cdots \lambda_m^k a_m q_m$$

Now, generalize: FIND n e'vectors/values

Suppose $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|$

let $V_0 = \begin{bmatrix} | & | & & | \\ v_1^0 & v_2^0 & \cdots & v_n^0 \\ | & | & & | \end{bmatrix}$

$\xleftarrow{\quad} n \xrightarrow{\quad}$

$V^k = A^k V_0$ where

$v_j = a_{1j} q_1 + a_{2j} q_2 + \cdots a_{mj} q_m$

$1 \leq j \leq n$

So $v_j^k = \lambda_1^k a_{1j} q_1 + \cdots \lambda_m^k a_{mj} q_m$

let $\hat{Q} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$ $m \times n$

and $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_m \\ | & & | \end{bmatrix}$ $m \times m$

$V^k = A^k V^0 = Q \Lambda^k Q^T V^0 =$

$= \hat{Q} \hat{\Lambda}^k \hat{Q}^T V^0 + O(|\lambda_{n+1}|^2)$

as $k \to \infty$

So $\hat{Q}$ has the n evectors

and $\hat{\Lambda}$ has the n e'values

$\hat{Q}^T V^0$ is non-singular

and we can iterate to extract

$\hat{Q}$, $\hat{\lambda}^k$:

ALGORITHM

Pick $\hat{Q}^0 \in \mathbb{R}^{m \times n}$

w/ $\perp$ columns

for $k = 1, 2, \dots$

$Z = A\hat{Q}^{k-1}$

$\hat{Q}^k \hat{R}^k = Z$ (a reduced factorization of Z)

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