THE POWER METHOD

Assume that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_m|$$

are the eigenvalues of $A \in \mathbb{C}^{m \times m}$

The spectral gap between λ_1 & the rest of the spectrum is essential for this method to work properly.

The matrix does not have to be non-defective, but for easy of presentation we'll assume this is the case.

Let $\{q_1, q_2, \dots, q_m\}$ be the e'vectors associated with $\lambda_1, \lambda_2, \dots, \lambda_m$.

Hence (#) $A q_i = \lambda_i q_i \quad 1 \le i \le m$

Let x_k be a set of $\mathbb{C}^m$ vectors where x_0 is a known initial guess.

Write $x_0 = \sum_{i=1}^{m} a_i q_i$, then let

$$x_1 = A x_0$$

$$x_2 = A x_1$$

$$\vdots$$

$$x_k = A^k x_0$$

(we'll fold into the q_i's the coefficients a_i: that is let $q_i \equiv a_i q_i$ from now on)

$$x_k = \lambda_1^k q_1 + \lambda_2^k q_2 \cdots + \lambda_m^k q_m$$

(using ($\ddagger$)).

Factoring λ_1^k

$$x_k = \lambda_1^k \left[q_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k q_2 + \cdots \left(\frac{\lambda_m}{\lambda_1}\right)^k q_m \right]$$

since $|\lambda_1| > |\lambda_i|$ $i=2,\dots,m$

then $\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ as $k \to \infty$

So x_0 will be "changing direction" and lining up with the direction of q_1

For finite k:

$$(\star) \quad x_k = \lambda_1^k \left[q_1 + \varepsilon_k \right]$$

where $e_k = \mathcal{O}\left[\left(\frac{\lambda_2}{\lambda_1}\right)^k\right]$ //

Implementation

Write an iteration process ... however, we need to be careful to keep the iterates balanced in magnitude. One way to do this is to take ratios ... of what? Let's formally rewrite ($\star$) as

$$\Theta(x_k) = \lambda_1^k \left[\Theta(q_1) + \Theta(\varepsilon_k)\right]$$

A reasonable choice for Θ is a norm:

$$r_k = \frac{\Theta(x_{k+1})}{\Theta(x_k)} = \lambda_1 \left[\frac{\Theta(q_1) + \Theta(\varepsilon_k)}{\Theta(q_1) + \Theta(\varepsilon_k)}\right]$$

So $r_k \to \lambda_1$ as $k \to \infty$

ALGORITHM:

provide x_0, an initial guess

for $k = 1, 2, \ldots$

$$y = Ax$$

$$r = \phi(y)/\phi(x)$$

$$x = y/\|y\|$$

output: $r \sim \lambda_1$

$x \sim q_1$

Relative Error: since $\lim_{k\to\infty} r_k = \lambda_1$

then $$\frac{r_k - \lambda_1}{\lambda_1} = \left(\frac{\lambda_2}{\lambda_1}\right)^k c_k$$

c_k is some $O(1)$ constant

which forms a bounded sequence.

Inverse Power Method

(Shifted Power Method)

Power Method can be used to find smallest eigenvalue

$$|\lambda_m^{-1}| > |\lambda_{m-1}^{-1}| \geq |\lambda_{m-2}^{-1}| \cdots \geq |\lambda_1^{-1}| > 0$$

(We know that if λ is an e'value of A and A is nonsingular then λ^{-1} is an e'value of A^{-1}

So power method could be formulated so that

$$x_k = (A^{-1})^k x_0$$

will then converge

to λ_m.

We don't know, or cannot safely compute A^{-1} so we introduce a SHIFT

The trick is to propose

$$\hat{A} = (A - \mu I) \text{ where } \mu \in \mathbb{C}$$

Then do power method on $\hat{A}$:

$$x_k = \hat{A}^k x_0.$$

So long as μ is reasonable guess the iteration process will yield λ_m & q_m

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