APPROXIMATING EIGENVALUES/VECTORS NUMERICALLY:

The fundamental difficulty with computing eigenvalue problems on a finite precision machine:

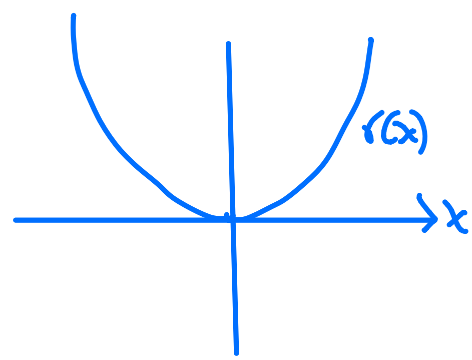
Finding roots $p(\lambda) = 0$,
the characteristic polynomial.

For $m \geq 5$ finding roots cannot be done analytically.

$\Rightarrow$ ALL eigenvalue solvers are ITERATIVE

The Rayleigh Quotient: Suppose A is real symmetric $A = A^T \in \mathbb{R}^{n \times n}$

$$r(x) = \frac{x^T A x}{x^T x}$$



Note: if $Ax = \lambda x$

$$r(x) = \frac{x^T \lambda x}{x^T x} = \lambda$$

$$\frac{\partial r}{\partial x_j} = (\nabla r)_j = \frac{\frac{\partial}{\partial x_j}(x^T A x)}{x^T x} - \frac{(x^T A x)\frac{\partial}{\partial x_j}(x^T x)}{(x^T x)^2}$$

$$= \frac{2(Ax)_j}{x^T x} - \frac{(x^T A x)\, 2x_j}{(x^T x)^2} = \frac{2}{x^T x}(Ax - r(x)x)_j$$

If set $\nabla r = 0$

if $x = 0$ $r(x)$ is arbitrary

if $x \neq 0$ $\begin{cases} \text{then } x \text{ is e'vector} \\ \text{then } \lambda = r(x) \end{cases}$

This suggests a procedure for finding λ (and x)

If A has $\lambda_{min} = \lambda_n \leq \lambda_{n-1} \leq \cdots < \lambda_1 = \lambda_{max}$

$$\therefore \lambda_{max} = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{|x|=1} x^T A x$$

Could also be finding

$$\lambda_{min} = \min_{x \neq 0} \frac{x^T A x}{x^T x} = \min_{|x|=1} x^T A x$$

Algorithm (finds the largest eigenvalue):

Start with x^0 initial guess

$$(\star) \qquad \lambda^{k+1} = \frac{\langle Ax^k, x^k \rangle}{\langle x^k, x^k \rangle} \qquad k=0,1,\ldots$$

where $\langle u,v \rangle = u^T v$ //

<u>Note:</u> if $x^0 = \sum_{j=1}^{m} \alpha_j q_j$

q_j are e'vectors of A, then

$x^k = A^k x^0 \quad \therefore$ substituting in $(\star)$:

$$\lambda^{k+1} = \frac{\langle A^{k+1} x^0, A^k x^0 \rangle}{\langle A^k x^0, A^k x^0 \rangle}$$

$$= \frac{\sum_{j=1}^{m} |\alpha_j|^2 \lambda_j^{2k+1}}{\sum_{j=1}^{m} |\alpha_j|^2 \lambda_j^{2k}}$$

So

$$\lambda^{k+1} = \frac{\sum_{j=1}^{m} |\alpha_j|^2 \left(\lambda_j/\lambda_1\right)^{2k+1} \lambda_1^{2k+1}}{\sum_{j=1}^{m} |\alpha_j|^2 \lambda_1^{2k} \left(\lambda_j/\lambda_1\right)^{2k}}$$

$$\lambda^{(k+1)} = \lambda_1 \left[1 + O\left(\left(\lambda_2/\lambda_1\right)^{2k}\right) \right]$$

Gives the largest e'value. Also gives the largest evector (see later) //

The Power Method Used to estimate the largest eigenvalue & its corresponding eigenvector. Require

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \ldots \geq |\lambda_m|$$

To introduce the method we'll presume that A is non-defective.