LECTURE 24 EIGENVALUE PROBLEMS

Let $A \in \mathbb{C}^{m \times m}$

$x \in \mathbb{C}^m$ is an eigenvector

if $Ax = \lambda x$ $(\exists)$

λ is e'value. $\lambda \in \mathbb{C}$

There are many specialized numerical methods for finding e'vectors & e'values. They are all iterative:

- Arnoldi
- Lanczos
- QR
- Power Method, Rayleigh Method
- Monte Carlo

SOME BACKGROUND:

Write (*) $M = A - \lambda I$

$Mx = 0$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ λ are to be x found.

If you know λ, find x in the null(M). To find λ:

Require $\det(M) = 0$ for $x \neq 0$ in $Mx = 0$

$\det(M) = p(\lambda) = 0$ The Characteristic Equation

$\lambda_1, \lambda_2 \dots, \lambda_m$ are the (complex) roots of characteristic eq.

the λ's don't all have to be unique.

Some other facts:

$$\det(A) = \prod_{j=1}^{m} \lambda_j \qquad tr(A) = \sum_{j=1}^{m} \lambda_j$$

def: **Defective Matrix:** if $A \in \mathbb{C}^{m \times m}$ is defective, A has less than m (L I) eigenvectors. In fact

A is non-defective iff

$$A = X \Lambda X^{-1}$$

X has m cols w/ eigenfunctions of A

and $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix}$ //

// def: Eigenspace E_λ is spanned by all of the eigenvectors associated with a particular λ:

$$E_\lambda \subseteq \mathbb{C}^m$$

The eigenspace is invariant: $A E_\lambda \subseteq E_\lambda$

$\dim(E_\lambda)$ is the total number of e'vectors

i.e. $\dim(\text{null}(A - \lambda I))$, for some λ.

The geometric multiplicity of λ is $\dim(E_\lambda)$

The algebraic multiplicity of λ is the multiplicity of the particular root λ in the characteristic polynomial $p(\lambda) = 0$. //

THREE SPECIAL SQUARE MATRICES:

Unitary Matrix: $A^{-1} = A^*$

Hermitian Matrix: $A = A^*$

Normal Matrix $AA^* = A^*A$

Hermitian Matrices have m eigenvalues, they are real, and m eigenvectors: if A is Hermitian

$$A = Q \Lambda Q^* \quad \text{(this is also an SVD)}$$

i.e. $AQ = Q\Lambda$

Q $\perp$, with normalized e'vectors of A

There's a family of matrices which include Hermitian, circulant, unitary matrices that are said to be UNITARILY DIAGONALIZABLE (UD): that is B is UD if

$$B = Q \Lambda Q^*$$

An important class of UD matrices are normal matrices.

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EIGENVALUE REVEALING FACTORIZATIONS

These are factorizations that make the eigenvalues explicit:

* **Diagonalization**: $A = X \Lambda X^{-1}$, applicable iff A is non-defective.

* **UD** : $A = Q \Lambda Q^*$ iff A is normal

* **SCHUR FACTORIZATION**:

$$A = QTQ^*$$ always exists!

T is a triangular matrix with e'values of A along its diagonal.

THE SCHUR FACTORIZATION

Take $A \in \mathbb{C}^{m \times m}$ want to find

$$A = QTQ^*$$

T is as above & Q $\perp$,

It's a recurrent factorization. Let's illustracte the first step:

Suppose we know (u, λ), the eigenpair. For simplicity, normalize u: i.e. $\|u\| = 1$

Obviously, $Au = \lambda u$.

FIRST SWEEP:

Construct a matrix $K \in \mathbb{C}^{m \times (m-1)}$ with orthonormal columns s.t. $K^* u = 0$

Let

$$Q = [u\ K] = \begin{array}{cc} \left[\begin{array}{c|c} \, u \, & \qquad K \qquad \end{array}\right] & \updownarrow m \\ \underleftrightarrow{\quad m \quad} & \end{array}$$

Q is unitary. Compute

$$Q^* A Q = [u\ K]^* A\, [u\ K]$$

$$= \begin{bmatrix} u^* A u & u^* A K \\ K^* A u & K^* A K \end{bmatrix} = \begin{bmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{bmatrix}$$

The (1,1) entry $u^* A u = u^* \lambda u = \lambda \|u\|^2 = \lambda$

The (2,1) entry $K^* A u = \lambda K^* u = 0$

$\dim (K^* A u) \in \mathbb{C}^{(m-1)\times 1}$

Nothing specific can be said about (1,2) and (2,2).

let $t_1^* \equiv u^* A K \in \mathbb{C}^{1\times(m-1)}$

let $A_1 \equiv K^* A K \in \mathbb{C}^{(m-1)\times(m-1)}$

$$A = [u\ K] \begin{bmatrix} \lambda & [\quad t_1^* \quad] \\ \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} \\ \quad A_1 \quad \\ \\ \end{bmatrix} \end{bmatrix} [u\ K]^*$$

At this point you can imagine that we can then make a similar factorization of $A_1 \in \mathbb{C}^{(m-1)\times(m-1)}$ to get $A_1 = [u_2 K_2] \begin{bmatrix} \lambda_2 & [\ t_2^* \ ] \\ \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} A_2 \end{bmatrix} \end{bmatrix} [u_2 K_2]^*$

and so on till we get

$$A = Q \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & 0 & & \ddots \\ & & & & \lambda_m \end{bmatrix} Q^* = QTQ^*$$

by a recurrent process.

But let's examine further

$$A = [u\ K] \begin{bmatrix} \lambda & t_1^* \\ 0 & A_1 \end{bmatrix} [u\ K]^*$$

let $A_1 = Q_1 T_1 Q_1^* \in \mathbb{C}^{(m-1)\times(m-1)}$

$$A = [u\ k] \begin{bmatrix} \lambda & t_1^* \\ 0 & Q_1 T_1 Q_1^* \end{bmatrix} [u\ K]^*$$

which can be written as

$$A = [u\ K] \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} \lambda & t_1^* Q_1 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1^* \end{bmatrix} \begin{bmatrix} u^* \\ K^* \end{bmatrix}$$

$$A = \underbrace{[u\ K\ Q_1]}_{Q} \underbrace{\begin{bmatrix} \lambda & t_1 Q_1 \\ 0 & T_1 \end{bmatrix}}_{T} \underbrace{[u\ K\ Q_1]^*}_{Q^*} \quad //$$

Rmk: A Schur factorization will reveal unitary or diagonal factorizations (the latter if A is non-defective)

Rmk: The recurrent process does not "overwrite" previously found e'values.

Rmk: if $A = QTQ^*$ in

$$(A - \lambda I)Q = 0 \quad \text{we get}$$

$$(QTQ^* - \lambda I)Q$$

$$= QT - \lambda I Q = (T - \lambda I)Q = 0$$

hence, A and T must have the same eigenvalues.

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