<u>Thm:</u> Let CG be applied to an SPD matrix A to solve $A\hat{x}=b$, where A and b are known.

Let $\text{cond}(A) = k(A)$. In $\|A\|_2 = \frac{|\sigma_{largest}|}{|\sigma_{smallest}|}$

$A \in \mathbb{R}^{m\times m}$, $b \in \mathbb{R}^m$.

Also let error at iterate n be $e_n = x_n - \hat{x}$.

Also for $f \in \mathbb{C}^m$, let $\|f\|_A = \sqrt{f^T A f}$

the A-norm:

For CG

$$\|e_n\|_A \leq \|e_{n-1}\|_A$$

and $e_n = 0$ is achieved for some $n \leq m$

and $\hat{x}$ is the unique minimizer of

$$f(x) = \frac{1}{2}x^T A x - x^T b$$

//

Thm: Rate of Convergence:

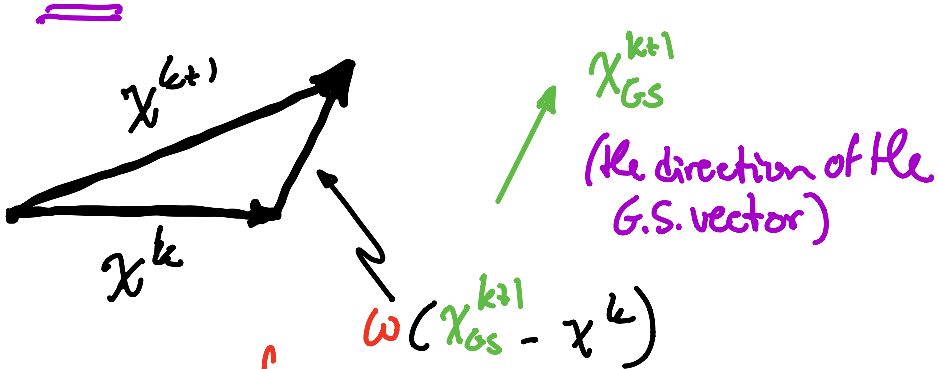
$$\frac{\|e_n\|_A}{\|e_0\|_A} \leq 2\left(\frac{\sqrt{k}-1}{\sqrt{k}+1}\right)^n.$$ In fact, for

$k \to \infty$,

$$\frac{\|e_n\|_A}{\|e_0\|_A} \sim 1 - \frac{2}{\sqrt{k}}.$$

Which implies that for large k, convergence to some tolerance is expected in $O(\sqrt{k})$ iterations. //

Rmk: Let's revisit SOR: $x^{k+1} = x^k + \omega(x_{GS}^{k+1} - x_k)$.



$$0<\omega<2 \begin{cases} \omega > 1 & \text{overrelaxation} \\ \omega < 1 & \text{underrelaxation} \\ \omega = 1 & \text{G.S.} \end{cases}$$

SOR is also a line search //