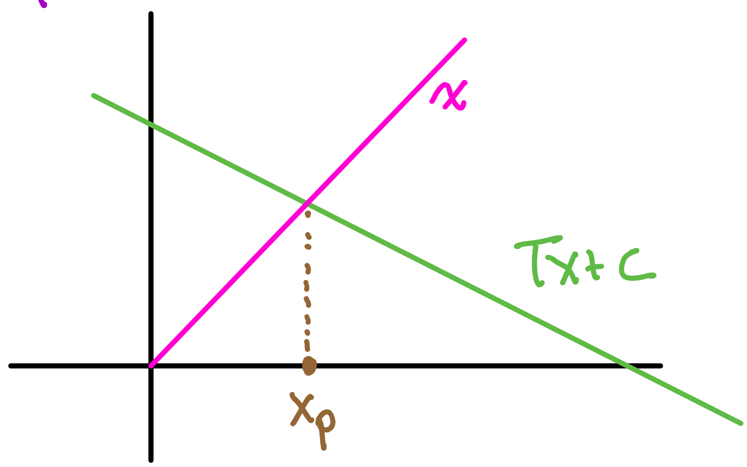
We saw that recasting $Ax-b=0$, the root finding problem, as a fixed point problem $x = Tx + c$



Makes $Ax_p - b = 0$ to $x_p = Tx_p + c$

We construct T (from A) by choosing $\|T\| < 1$ for a unique fixed point x_p

CONVERGENCE OF ITERATIVE METHODS:

$$x_{k+1} = Tx_k + c \quad \begin{cases} \text{Jacobi} \\ \text{SOR} \\ \text{Gauss Seidel (G.S.)} \end{cases}$$

let $e_k = x - x_k$

$$\therefore\ x_{k+1} - x = T(x_k - x) + c - c$$

$$e_{k+1} = T e_k$$

$$e_k = T e_{k-1}$$

$$\vdots$$

So $e_n = T^n e_0 \quad n = 1, 2 \ldots$

The sequence $x_0, x_1 \ldots x_k \ldots$ will converge to x, as $k \to \infty$, as long as T is a convergent matrix

In that case

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} T^n e_0 = 0 \quad //$$

We can also use spectral methods to see this (This is also useful analytical technique to explore quantitative differences in the norm of T: For simplicity assume

$T \in \mathbb{C}^{m \times m}$ full rank (rank $= m$)

let $\{v_s\}_{s=1}^{m}$ be the m (LI) eigenvectors of T w/ e'values λ_s

then we can express:

$$e_0 = \sum_{s=1}^{m} c_s v_s \qquad c_s \text{ are coefficients}$$

$$e_1 = \sum_{s=1}^{m} c_s T v_s = \sum c_s \lambda_s v_s$$

$$\vdots$$

$$e_k = \sum c_s \lambda_s^k v_s$$

So we see that

$$\lim_{k \to \infty} e_k = 0 \quad \text{if } \rho(T) < 1$$

i.e. $\max |\lambda_s|_{s=1}^{m} < 1$ //

JACOBI:

$$T = D^{-1}(L+U)$$

$$x_{k+1} = T x_k + c \Rightarrow (T - \lambda_s I) v_s = 0$$

leads to a characteristic polynomial

$P(\lambda_s) = 0$ which is

$$\det\left[\lambda I - D^{-1}(L+U)\right] = 0$$

$$= \det D^{-1} \det(\lambda D - L - U) = 0$$

where $\det D^{-1} = \dfrac{1}{\det D} = \dfrac{1}{\prod_{i=1}^{m} a_{ii}} \neq 0$

$A = \{a_{ij}\}$

$\therefore$ factor this out

$$\det(\lambda D - L - U) = 0$$

require $|\lambda| < 1 \Rightarrow \|D\| > \|L+U\|$

where $\det(\lambda I - T) = 0$

i.e. λ are the e'vs of T, not A.

$\therefore$ Jacobi should converge for diagonally dominant A //

SOR (includes GS)

To simplify notation let's factor out the D of $Ax = b$

$$\underbrace{D^{-1}A}_{B}x = D^{-1}b \qquad B = D^{-1}A$$

$$Bx = c \qquad c = D^{-1}b$$

where $A = D + \tilde{L} + \tilde{U}$

$$\Rightarrow B = I + L + U$$

Solve $Bx = c$ via SOR:

$$x_{k+1} = x_k - \omega\left((I+U)x_k + Lx_{k+1} - c\right)$$

Eliminating U:

$$(\star) \quad (I + \omega L)(x_{k+1} - x_k) = -\omega(Bx_k - c)$$

let $\varepsilon_k \equiv Bx_k - c$

$\therefore (\star)$ can be written as

$$\varepsilon_{k+1} = Bx_k - c$$

$$B^{-1}(\varepsilon_{k+1} - \varepsilon_k) = x_{k+1} - x_k \quad \therefore (A)$$

becomes

$$(I + \omega L)B^{-1}(\varepsilon_{k+1} - \varepsilon_k) = -\omega\varepsilon_k$$

finally:

$$\varepsilon_{k+1} = [\underbrace{I - \omega B(I + \omega L)^{-1}}_{I - K(\omega)}]\varepsilon_k$$

$$\varepsilon_{k+1} = [I - K(\omega)]\varepsilon_k$$

For convergence we require

$$\| I - K(\omega) \| < 1$$

Reverting back to $Ax = b$

we have $\det[(\lambda + \omega - 1)D - \lambda\omega\tilde{L} - \omega\tilde{U}] = 0$

In principle we could tune ω so that $\| I - K(\omega) \| < 1$ and as small as possible.

let's focus on GS SOR with $\omega = 1$

$$\det[\lambda D - \lambda \tilde{L} - \tilde{U}] = 0$$

$$\det[\lambda(D - \tilde{L}) - \tilde{U}] = 0$$

so for $|\lambda| < 1$

want $\|(D - \tilde{L})\| > \|\tilde{U}\|$

Which can be helped by reorganizing rows in $Ax = b$

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