FIXED POINT PROBLEM

def: A fixed point p is the value for which

$$g(p) = p.$$

We'll restrict our attention to $g(x)$ functions on the real line $x \in \mathbb{R}$.

Rmk: The fixed point problem is related to the root finding problem $f(p) = 0$ by noticing that

$$f(p) \equiv g(p) - p = 0$$

$\therefore$ in principle, one can turn many fixed point problems into root-finding problems and viceversa. //

KEY ISSUES IN FIXED POINT PROBLEMS:

- What $g(x)$ have fixed points?
- When is the fixed point unique?
- How do we estimate a fixed point?

ex) $g(x) = x$ for $0 \leq x \leq 1$, say, has fixed points $p \in [0,1]$.

ex) $g(x) = x^2$ for $0 \leq x \leq 1$, say, has $p = 0, 1$ for fixed points.

Thm: (EXISTENCE & UNIQUENESS) If $g \in C[a,b]$, such that $g(x) \in [a,b]$ $\forall x \in [a,b]$, then $g(x)$ has one or many fixed points in $x \in [a,b]$.

Spse, in addition, that $g'(x)$ exists on $[a,b]$ and $\exists$ a constant $0 < k < 1$ s.t.

$$|g'(x)| \leq k \quad \forall x \in (a,b)$$

$\Rightarrow$ $g(p) = p$ is unique //

Pf: (existence) if $g(a) = a$ or $g(b) = b$ then fixed point exists. Suppose not, then it must be true that $g(a) > a$ and $g(b) < b$. Define

$h(x) = g(x) - x$. Then h is $C[a,b]$ and

$h(a) = g(a) - a > 0 \quad h(b) = g(b) - b < 0$.

The INTERMEDIATE VALUE THEOREM implies that $\exists\, p \in (a,b)$ for which $h(p) = 0$ $\therefore$ $g(p) - p = 0$ (fixed points exist).

Uniqueness: S'pse in addition $|g'(x)| \le k < 1$ $\forall x \in [a,b]$. S'pse that p and q are both fixed points in $[a,b]$, with $p \neq q$. By the mean value theorem $\exists$ a number between p and q such that

$$\frac{g(p) - g(q)}{p-q} = g'(\xi)$$

for some $\xi \in [a,b]$. Then

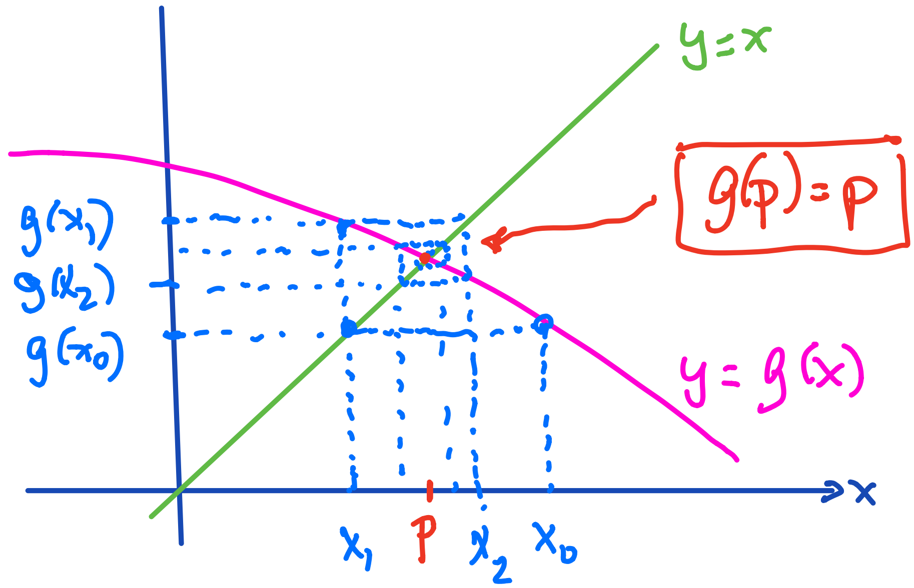
$$|p-q| = |g(p) - g(q)| = |g'(\xi)||p-q|$$
$$\le k|p-q| < |p-q| \Rightarrow \Leftarrow$$

$\therefore$ $p = q$. //

Rmk: The theorem is one way. There is no converse statement.

Rmk: fixed points may exist under less restrictive conditions. //

How do we calculate a fixed point?



Iterative Strategy:

x_0 initial guess

$$x_{n+1} = g(x_n) \qquad n=0,1\ldots$$

This procedure will find a unique fixed point if the existence & uniqueness conditions are met, regardless of the starting guess x_0.

Thm (Fixed Point Iteration): Let $g \in C[a,b]$ and $g(x) \in [a,b]$ $\forall x \in [a,b]$. Sp'se g' is $C(a,b)$, with

$$|g'(x)| \le k < 1 \quad \forall x \in (a,b).$$

If $g'(p) \neq 0$ then, for any p_0 (initial guess) in $[a,b]$, the sequence

$$p_n = g(p_{n-1}) \qquad n = 1, 2, \ldots$$

converges linearly to the unique fixed point in $[a,b]$.

<u>Proof</u>: The FP Theorem says that $\{p_n\}_{n=0}^{\infty} \to p$.

Since g' exists on (a,b) we can apply the Mean value theorem to g to show that for any n, integer,

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p)$$

$$\therefore \quad \lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n\to\infty} g'(\xi_n) = g'(p)$$

$$\text{and} \quad \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$$

$\therefore$ the fixed point problem converges linearly if $g'(p) \neq 0$: $|p_{n+1} - p| \le k |p_n - p|^1$

//

Under certain circumstances we can get better than linear convergence: e.g. $g'(p) = 0$ and $g'' \in C(a,b)$ and strictly bounded by M constant in the interval $[a,b]$ then

$$p_n = g(p_{n-1}),\ n \geq 1 \text{ will converge}$$

$$|p_{n+1} - p| \leq \frac{M}{2} |p_n - p|^2$$

converges quadratically

<u>Corollary of fixed point iteration theorem</u>

If g satisfies the FP iteration theorem, then the bounds on the error involved in using p_n to approximate p are given by

$$|p - p_n| \leq k^n \max(p_0 - a, b - p_0)$$

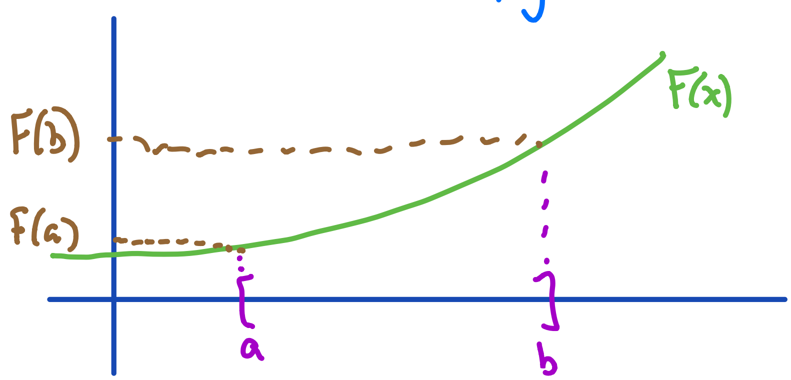
$$|p - p_n| \leq \frac{k^n}{1-k} |p_1 - p_0| \quad n = 1, 2, \ldots$$

Rmk: The fixed point theorem is the simplest version of a more general result, called "the contraction mapping theorem".

def: Let $C \subseteq \mathbb{R}$ and $F: C \to C$. We say that F is contractive if $\exists$ a constant $\lambda < 1$ s.t.

$$|F(x) - F(y)| \leq \lambda |x - y|$$

$$\forall x, y \in C$$

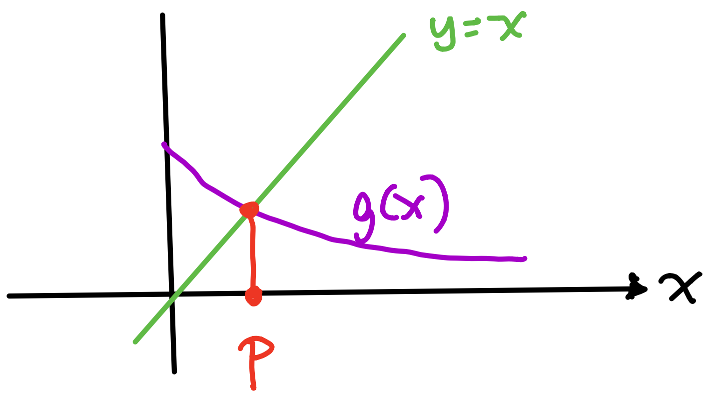


in $[a,b]$ the values of F are not bigger than $[a,b]$

ex) Consider what happens if we find the root(s) of $e^{-x} - x = 0$ using fixed point iteration:

(a) $x = e^{-x} = g(x)$

then $|g'(x)| < 1$ for $x > 0$



$x_{n+1} = e^{-x_n}$

for $x_0 > 0$ will converge to the unique fixed point

Consider instead the equivalent

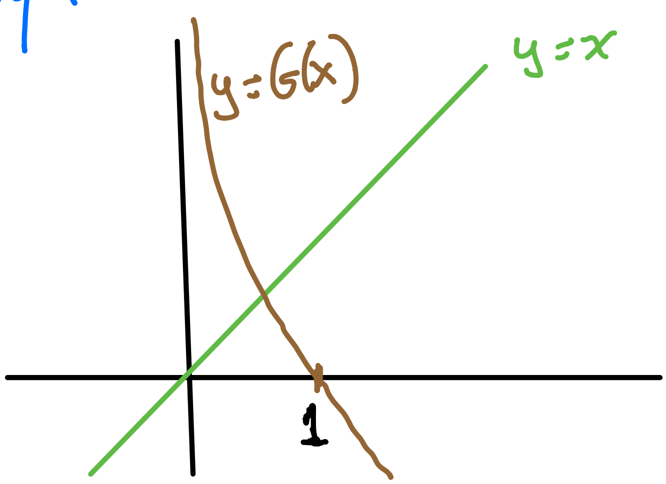
problem:

$$x = \ln\left(1/x\right) = G(x)$$

Take $x > 0$,

$$|G'(x)| = \frac{1}{|x|}$$

for small $|x|$ the derivative is not bounded by 1: there still is a fixed point, but the theorem does not apply. Furthermore, try this out:



(A) $\begin{cases} x_{n+1} = \ln(1/x_n) \\ x_0 = 0.5 \end{cases}$

Do 2 or 3 steps of iteration to see the trend in the sequence.

Also, take

(B) $\begin{cases} x_{n+1} = \ln(1/x_n) \\ x_0 = 10 \end{cases}$

Do a few steps to see what happens:

	A	B
$n=0$	0.5000	10
$n=1$	0.6065	⋮
$n=2$	0.5452	⋮ junk
$n=3$	0.5797	⋮
⋮	⋮	⋮
$n=10$	0.5669	⋮

In case B we started in neighborhood where $|G'(x)| < 1$, but once we left that neighborhood, this was no longer the case. In any event, even the first iteration lead to $x < 0$, nonsense //

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A NOTE ON SHOWING CONVERGENCE NUMERICALLY:

In fixed point problems we saw that

$$|p_{n+1} - p| < C|p_n - p|^{\alpha}$$

where α was <u>at least</u> 1.

We will want to show that our code/algorithm obeys the right

Convergence rate: So we produce

CONVERGENCE PLOTS

Suppose a method has a convergence rate

$$\|x_{n+1} - x\| < C\|x_n - x\|^\alpha$$

and we know α. We will seek evidence this is the case by tabulating

$x_0 - x$	$n=0$
$x_1 - x$	$n=1$
$\vdots$	$\vdots$
$x_n - x$	n
$\vdots$	$\vdots$

let $y_n := \|x_n - x\|$

then $y_{n+1} < C\, y_n^\alpha$

so $\log y_{n+1} < \log C + \alpha \log y_n$

So plot

You should see that, provided $n > n_0$ that the code delivers an error that when plotted in loglog, drops with a slope α