Pivoting may still fail: take same problem as before but multiply E_1 by a large number, say by 10^4:

$$E_1: \quad 30.00x_1 + 591400x_2 = 591700$$

$$E_2: \quad 5.291x_1 - 6.130x_2 = 46.78,$$ using 4 digit precision:

let $m_{21} = \dfrac{5.291}{30.00} = 0.1764$, then $E_2 - m_{21}E_1 = E_2$:

$$\begin{cases} 30.00x_1 + 591400x_2 = 591700 \\ \qquad\qquad -104300x_2 = -104400 \end{cases}$$

$x_1 = -10 \qquad x_2 = 1.001$ (wrong!)

Problem is the disparate sizes of quantities...

A simple fix is to use total or partial scaling:

The simplest is "row scaling"

$$Ax = b$$

multiply E_i by r_i, $i = 1, 2, .., m$, or

$$D_1 A x = D_1 b \qquad D_1 = \begin{bmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_m \end{bmatrix}$$

let $C = D_1 A$

choose r_i so that $C = \{c_{ij}\}$ satisfies

$$\max_j |c_{ij}| \sim 1 \qquad i = 1, 2 \ldots, m$$

so $$\frac{1}{r_i} = \max_j |a_{ij}| \qquad i = 1, 2, \ldots, m$$

$$\therefore \quad c_{ij} = r_i a_{ij} \qquad i = 1, 2 \ldots, m \ldots$$

This might not be helpful (need to check the condition number of $Cx = D_1 b$ (division) and check each row for loss precision in the subtraction stage

Scaling can be done once, before the GE, or done as GE proceeds (in both cases, you still need to worry about stability)

Scaling $Ax = b$ is usually a good idea regardless of numerical linear algebra solver strategy used, i.e. GE, QR, SVD, etc

Rmk: Not every $Ax=b$ can be LU-solved.

ex) $$4x_2 = 8$$
$$4x_1 + 2x_2 = 17$$

$$A = \begin{bmatrix} 0 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$u_{11} = 0$$

$$l_{21} u_{11} = 4 \quad \text{(not possible)!}$$

However,

$$4x_1 + 2x_2 = 17$$
$$4x_2 = 8$$

can be LU decomposed:

$$\begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix} \quad \text{ok!}$$

Rmk: For $Ax=b$, if A is DIAGONALLY-DOMINANT GE can be performed w/o row exchanges.

$A = \{a_{ij}\}$ is DIAGONALLY-DOMINANT if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \qquad i,j = 0,1,\dots,m$$

Lecture 23 Cholesky Factorization

It's an LU factorization, applicable to

(HPD) HERMITIAN POSITIVE DEFINITIVE MATRICES if $A \in \mathbb{C}^{m\times m}$

If $A \in \mathbb{R}^{m\times m}$ HPD is SPD (Symmetric Positive Definite matrix.

def: $A \in \mathbb{R}^{m\times m}$ is symmetric if $A = A^T$ $(a_{ij} = a_{ji})$

def: $A \in \mathbb{C}^{m\times m}$ is Hermitian if $A^* = A$ $(a_{ij} = \overline{a_{ji}})$

(note: this means that the diagonal entries must be real)

If A is Hermitian then

$$x^* A y = \overline{y^* A x} \quad \forall x, y \in \mathbb{C}^m$$

holds. In particular, if $y = x$ and $x \in \mathbb{C}^m$

$$x^* A x \quad \text{is real}$$

If, in addition,

$x^*Ax > 0 \quad \forall x \neq \emptyset$

A is HPD

SOME PROPERTIES OF HERMITIAN & HPD :

* The e'values of HPD are **> 0, and Real**
(in fact if you can show that A has all positive real e'values and is symmetric $\Rightarrow$ A is HPD)
* The e'vectors of Hermitian matrix corresponding to distinct e'values are $\perp$.
* If A has m e'values then $A = Q \Lambda Q^*$ and e'vectors are **$\perp$ and REAL** //

Solving $Ax=b$, using Cholesky

Cholesky is a special variant of LU, i.e. you get the same decomposition. Note the LU decomposition of an arbitrary A is not unique. However if A is symmetric the LU (Cholesky) will be unique. We'll see why in what follows:

Let $A \in \mathbb{C}^{m \times m}$ HPD

Perform 1st step of GE:

let $\alpha = \sqrt{a_{11}}$

$$A = \begin{bmatrix} a_{11} & [\,w^*\,] \\ [\,w\,] & K \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ [\,\frac{w}{\alpha}\,] & [\,I\,] \end{bmatrix} \underbrace{\begin{bmatrix} \alpha & [\,\frac{w^*}{\alpha}\,] \\ [\,0\,] & [\,K - \frac{ww^*}{\alpha^2}\,] \end{bmatrix}}$$

$$\begin{bmatrix} \alpha & [\,\frac{w^*}{\alpha}\,] \\ [\,0\,] & [\,K - \frac{ww^*}{\alpha^2}\,] \end{bmatrix} = \begin{bmatrix} 1 & \boxed{0} \\ [\,0\,] & [\,K - \frac{ww^*}{\alpha^2}\,] \end{bmatrix} \begin{bmatrix} \alpha & \boxed{\frac{w^*}{\alpha}} \\ [\,0\,] & [\,I\,] \end{bmatrix}$$

This submatrix is symmetric!

$$A = \begin{bmatrix} a_{11} & [\,w^*\,] \\ [\,w\,] & [\,K\,] \end{bmatrix} = \begin{bmatrix} \alpha & [\,0\,] \\ [\,\frac{w}{\alpha}\,] & [\,I\,] \end{bmatrix} \begin{bmatrix} 1 & \boxed{0} \\ [\,0\,] & [\,K - \frac{ww^*}{a_{11}}\,] \end{bmatrix} \begin{bmatrix} \alpha & \frac{w^*}{\alpha} \\ 0 & [\,I\,] \end{bmatrix}$$

Perform same decomposition on this submatrix and recursively repeat process m times. In the end the middle matrix will be $I_{m\times m}$ ∴ This generates a factorization: Cholesky factorization

$$A = R^*R = \underbrace{R_1^* R_2^* \cdots R_m^*}_{R^*} \underbrace{R_m R_{m-1} \cdots R_2 R_1}_{R}$$

$r_{jj} > 0$

Note: We assumed in (∗) that $a_{11} > 0$ and used this, or something similar for each decomposition stage. They must be > 0, since $K - ww^*/\alpha$ is positive definite, then we're OK.

Thm: Every HPD $A \in \mathbb{C}^{m\times m}$ has a unique Cholesky decomposition.

Pf: see algorithm.

Computational Complexity: for $A \in \mathbb{C}^{m\times m}$

Storage is $O(m^2)$ and the speed of algorithm is $O(m^3)$ flops.

Stability: $Ax = b$ Cholesky is backward stable.