Continuing the discussion on partial pivoting:

E_1: $0.003000x_1 + 59.14x_2 = 59.17$

E_2: $5.291x_1 - 6.130x_2 = 46.78$

has exact solutions to 4-digit rounding:

The exact answer was $x_1 = 10.00$, $x_2 = 1.000$. The GE with 4 digit rounding gave $x_1 = -10.00$ and $x_2 = 1.01$

The problem we encountered can be understood as follows: Take $Ax = b$ and start row reduction: (as in the above example) if $|a_{11}|$ is small compared to

$|a_{i1}| \Rightarrow \lambda \equiv m_{i1} = -\frac{a_{i1}}{a_{11}}$ is very large

then $(E_i + \lambda E_1) \to E_i$ The row operations leads to

$(a_{i2} + \lambda a_{12})x_2 + \cdots (a_{im} + \lambda a_{1m}) x_m = b_i + \lambda b_1$

$\therefore$ if $|\lambda|$ is large, you wind up adding a large multiple of E_1 to E_i $i \neq 1$.

Recall that we are assuming that $A \in \mathbb{C}^{m \times m}$ is full rank. GE turns the problem $Ax = b$ into an easier problem $Cx = p$ (equivalent) via row operations. If λ is large, we are getting two rows of C that are nearly linearly dependent. Hence C starts looking singular (in finite precision).

If λ is small, then no problem with roundoff. Hence the remedy is:

∴ Pivoting via row exchanges is performed as follows:

Find row index I s.t.

$$|a_{I1}| = \max_{1 \le i \le m} |a_{i1}|$$

& rewrite system $Ax = b$ so that the I^{th} equation becomes the first equation.

then $-\frac{a_{i1}}{a_{I1}}$ is small

& repeat.

Partial Pivoting is done via Permutation matrices. This avoids the slow and memory-intensive process of moving "equations" around in computer memory.

Permutation matrices exactly m bits of memory storage and provides us with a map of all row exchanges performed. //

The permutation matrix P: used to permute rows.

P has precisely 1 entry (1) in each row OR column

ex) $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Compute

(☆)

$$PA = P \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Get a different answer if you RIGHT multiply!

PROPERTIES: (1) PA permutes rows of A

(2) P^{-1} exists $P^{-1} = P^T$

(3) P can be stored in m-vector

ex) Take ☆ can be stored as (1,3,2)

ex) $A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 3 \end{bmatrix}$ no LU since $a_{11} = 0$!

Perform the following row ops:

$E_1 \leftarrow E_4$

$P = \begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}$ if applied $PA = \begin{bmatrix} 1&2&-1&3 \\ 1&1&-1&2 \\ 1&1&0&3 \\ 0&0&-1&1 \end{bmatrix}$

P is stored as (4,2,3,1)

SCALING:

Pivoting may still fail: (4 dig precision)

ex) $E_1: 30.00x_1 + 591400x_2 = 591700$

$E_2: 5.291x_1 - 6.130x_2 = 46.78$

$\lambda = \frac{5.291}{30} = 0.1264$

$30.00x_1 + 591400x_2 = 591700$

$-104300x_2 = -104400$

$\Rightarrow x_1 = -10.00 \quad x_2 = 1.001$

Wrong!