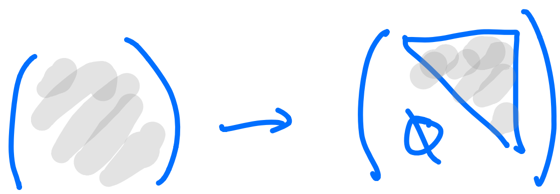
Lecture 20 GAUSSIAN ELIMINATION (GE)

Solve $Ax = b$ where $A \in \mathbb{C}^{m\times m}$, nonsingular
$x \in \mathbb{C}^m$, $b \in \mathbb{C}^m$

GE transforms problem into an upper triangular equivalent problem, which can be solved by back substitution.



A

$$\underbrace{L_{m-1} \cdots L_2 L_1}_{L^{-1}} A = U \qquad \therefore\ A = LU$$

We use the familiar ROW ECHELON operations to find U (and L)

As far as solving $x = A^{-1}b$, we note that

$$LUx = b \implies \begin{cases} y = Ux \\ Ly = b \end{cases}$$

Algorithm: (w/o pivoting)

4x4

$$\underset{A}{\begin{pmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{pmatrix}} \xrightarrow{L_1} \underset{L_1A}{\begin{pmatrix} \times & | & | & | \\ 0 & | & | & | \\ 0 & | & | & | \\ 0 & | & | & | \end{pmatrix}} \xrightarrow{L_2} \underset{L_2L_1A}{\begin{pmatrix} \times & \times & | & | \\ 0 & \times & | & | \\ 0 & 0 & | & | \\ 0 & 0 & | & | \end{pmatrix}} \xrightarrow{L_3} \underset{L_3L_2L_1A}{\begin{pmatrix} \times & \times & \times & | \\ 0 & \times & \times & | \\ 0 & 0 & \times & | \\ 0 & 0 & 0 & | \end{pmatrix}}$$

$$\xrightarrow{L_4} \underset{L_4L_3L_2L_1A}{\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & 0 & \times & \times \\ & & & \times \end{pmatrix}} = U$$

Rmk:

GS	$A = QR$	triangular orthogonalization
Householder	$A = QR$	orthogonal triangularization
LU	$A = LU$	triangular triangularization

ex)
$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 2&-2&3&-3&-20\\ 1&1&1&0&-2\\ 1&-1&4&3&4\end{array}\right) \xrightarrow[E_2=E_2-2E_1]{} \left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 0&0&-1&-1&-4\\ 1&1&1&0&-2\\ 1&-1&4&3&4\end{array}\right) \xrightarrow[E_3=E_3-E_1]{}$$

$$\left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 0&0&-1&-1&-4\\ 0&2&-1&1&6\\ 1&-1&4&3&4\end{array}\right) \xrightarrow[\substack{E_3\to E_2\\ E_2\to E_3}]{} \left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 0&2&-1&1&6\\ 0&0&-1&-1&-4\\ 1&-1&4&3&4\end{array}\right) \xrightarrow[E_4=E_4-E_1]{}$$

$$\left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 0&2&-1&1&6\\ 0&0&-1&-1&-4\\ 0&0&2&4&12\end{array}\right) \qquad (A|b) \to (L_1 A | L_1 b) \qquad \xrightarrow[E_4=E_4+2E_3]{}$$

$$\left(\begin{array}{cccc|c} 1&-1&2&-1&-8\\ 0&2&-1&1&6\\ 0&0&-1&-1&-4\\ 0&0&0&2&4\end{array}\right) \qquad (\underbrace{L_2 L_1}_{L^{-1}} A | L_2 L_1 b) \text{ so } L^{-1}A = U$$

Then we can back substitute from bottom to top:

$$2x_4 = 4 \Rightarrow x_4 = 2$$

$$-x_3 = x_4 - 4 \Rightarrow x_3 = 2$$

$$2x_2 = 6 + x_3 - x_4 \Rightarrow x_2 = 3$$

$$x_1 = x_2 - 2x_3 + x_4 - 8 \Rightarrow x_1 = -7$$

//

COMPUTATIONAL COMPLEXITY $Ax=b \Rightarrow (A|b)$

Storage 2D array $\mathbb{C}^{m,m+1}$
(since the "lower" portion of the array, where zeros go,
can provide all the storage you need for the
factors that compute the L matrix

Speed (measured in FLOPS) $\propto O(m^3)$

To see this:

GAUSSIAN ELIMINATION ALGORITHM

Solves $Ax=b$ $m\times m$

Input $(A|b) \equiv (a_{ij})$ $1\le i\le m$ $1\le j\le m+1$ AUGMENTED MATRIX
Output x_i $1\le i\le m$, or error message

(1) for $i=1:m-1$ % elimination process
  (2) Let p be smallest integer with $i\le p\le m$ & $a_{pi}\ne 0$
  if no integer p can be found: output('no unique solution') stop
  (3) if $p\ne i$ $E_p \leftrightarrow E_i$
  (4) for $j=i+1:m$
  (5) $m_{ji} = a_{ji}/a_{ii}$
  (6) $E_j - m_{ji}E_i \leftrightarrow E_j$

(7) if $a_{mm}=0$: output('no unique solution'), stop
(8) $x_m = a_{m,m+1}/a_{mm}$ % begin backsubstitute
(9) for $i=m-1:1$
  $x_i = \left(a_{i,m+1} - \sum_{j=i+1}^{m} a_{ij}x_j\right)/a_{ii}$

(10) output(x)
end

OPERATION COUNT

In pink are lines of interest.

Look at (5) & (6):

(5) $(m-i)$ divides

(6) multiplies results in $(m-i)(m-i+1)$

then $(m-i)(m-i+1)$ subtracts

So for (5) & (6), for each $i = 1, 2, \dots, m-1$

Mult/div $(m-i) + (m-i)(m-i+1) = (m-i)(m-i+2)$

adds/sub $(m-i)(m-i+1)$

$\therefore$ Mults/Div: $\sum_{i=1}^{m-1} (m-i)(m-i+2) = (m^2+2m)\sum_{i=1}^{m-1} 1$

$- 2(m+1)\sum_{i=1}^{m-1} i + \sum_{i=1}^{m-1} i^2 = \dfrac{2m^3+3m^2-5m}{6}$

Sub/Add: $\sum_{i=1}^{m-1} (m-i)(m-i+1) = (m^2+m)\sum_{i=1}^{m-1} 1$

$-(2m+1)\sum_{i=1}^{m-1} i + \sum_{i=1}^{m-1} i^2 = \dfrac{m^3-m}{3}$

(8) & (9): $1+(m-i)$ mults, $(m-i+1)$ adds for each summation term, then 1 sub & 1 divide

$\therefore \quad \dfrac{m^2+m}{2}$ (mults/divs) $+ \dfrac{m^2-m}{2}$ (adds/subs)

$$\therefore \underbrace{\frac{m^3}{3} + m^2 - \frac{m}{3}}_{\text{mults/divs}} + \underbrace{\frac{m^3}{3} + \frac{m^2}{2} - \frac{5m}{6}}_{\text{add/sub}} = O(m^3)$$

PIVOTING & SCALING: how to minimize the effect of round off errors in an ill-conditioned matrix problem?

Partial (row) Pivoting: row exchanges

Full (column/row) Pivoting: row & col exchanges

Motivating Example:

ex)
$$\left.\begin{aligned} 0.0001x + 1.00y &= 1.00 \\ 1.00x + 1.00y &= 2.00 \end{aligned}\right\} (\ddagger)$$

True solution (to 5 significant digits)

$x = 1.00010 \qquad y = 0.99990$

we used exact arithmetic & chopped to 5 sig digits

Solution of $(\ddagger)$ to 5-significant digits will be

$$-10000y = -10000 \Rightarrow y = 1.00$$

and $x = 0.00$ Pretty bad!

We did Gaussian elimination

We will do partial pivoting: all we do here is switch the rows. Proceed with Gaussian elimination:

$$1.00x + 1.00y = 2.00$$
$$0.0001x + 1.00y = 1.00$$

Perform GE (3 significant digit)

$$x = 1.00$$
$$y = 1.00$$

Great improvement!

ANALYSIS OF PROBLEM W/ ROUNOFF:

ex) $E_1 : \; 0.003000x_1 + 59.14x_2 = 59.17$

$E_2 : \; 5.291x_1 - 6.130x_2 = 46.78$

has exact solution, to 4-digit rounding:

$$x_1 = 10.00 \qquad x_2 = 1.00$$

$a_{11} = 0.003000$ is pivot & is small

$$m_{21} = \frac{5.291}{a_{11}} = 1763.6\overline{6} \text{ rounds } 1763.$$

$E_2 - m_{21}E_1 \rightarrow E_2$

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$-104300x_2 = -104400$$

instead of the more precise

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$-104309.37\overline{6}\,x_2 = -104309.37\overline{6}$$

∴ disparity in magnitudes m_{21}, b_1 and b_2 has introduced round off, but error has not propagated, yet.

Backward solution

$x_2 = 1.001$ close to $x_2 = 1.000$

but

$$x_1 = \frac{59.17 - (59.14)(1.001)}{0.003000} = -10.00$$