Lecture 18/19

Conditioning of Least Squares

Finding estimate to the solution of

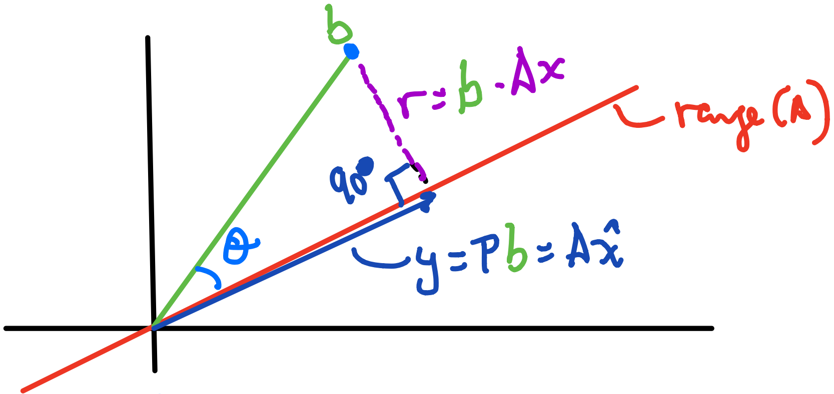
$$Ax=b \quad , \quad b\in\mathbb{C}^m$$

A is full rank $\in\mathbb{C}^{m\times n}$ $\quad m\geq n$

Assume $\|\cdot\| = \|\cdot\|_2$

$$\text{LSQ}\begin{cases}\text{Find } x\in\mathbb{C}^n \text{ st. } \|\underbrace{b-Ax}_{r}\| \text{ is minimized.}\\ \text{Call that } x=\hat{x}\end{cases}$$

The Solution $\hat{x}=A^+b$ $\qquad y=Pb=A\hat{x}$



$$\begin{cases}A^+\in\mathbb{C}^{n\times m}\\ P=AA^+\in\mathbb{C}^{m\times m}\end{cases}$$

Consider LSQ in light of conditioning, i.e. response to perturbations (lecture 18), response to algorithms (lecture 19)

Perturbations: to "model" A
to "data" b

Impact: $\hat{x}$ and $y = Pb$

Recall, for square matrix

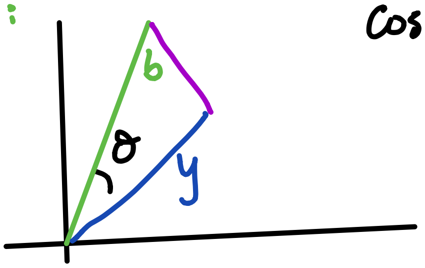
$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

for rectangular matrix

$$\kappa(A) = \|A\| \, \|A^{+}\| = \frac{\sigma_1}{\sigma_n} \quad \text{(in 2-norm)}$$

Model Sensitivity $1 \le \kappa(A) \le \infty$

Closeness of Fit:



$$\cos\theta = \frac{\|y\|}{\|b\|}$$

$0 \le \theta \le \frac{\pi}{2}$ (can choose different models A)

really good

really bad

How much does $\|y\|$ fall short of its possible maximum value:

what the data really is like.

$$\frac{\|P(b+\delta b)\|}{\|Pb\|} \sim \eta$$

what you expect

obtained from $b+\delta b$

$$\eta \equiv \frac{\|A\|\|\hat{x}\|}{\|y\|} = \frac{\|A\|\|\hat{x}\|}{\|A\hat{x}\|}$$

$$1 \leq \eta \leq \kappa(A)$$

good

Th'm: Let $b \in \mathbb{C}^m$ & $A \in \mathbb{C}^{m\times n}$ of full rank. The LSQ has the following 2-norm relative condition numbers:

		$Pb=y$	$\hat{x}$
exact, attained by certain δb	b	$\frac{1}{\cos\theta}$	$\frac{\kappa(A)}{\cos\theta}$
(upper bounds) on this row	A	$\frac{\kappa(A)}{\cos\theta}$	$\kappa(A) + \frac{\kappa^2(A)\tan\theta}{\eta}$

Special Case $m=n$ full rank $\Rightarrow \theta=0$

	x
b	$\kappa(A)/\eta$
A	$\kappa(A)$

Lecture 19

Download lecture19.mlx from the classnotes web page and follow along.

In this lecture we look at the stability of the various algorithms available to compute LSQ