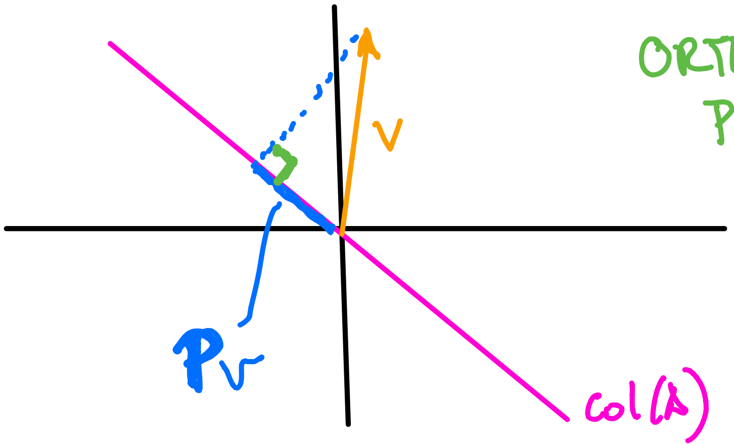
PART II (Projectors, least Squares)

PROJECTORS

PROJECTOR: Square matrix that satisfies

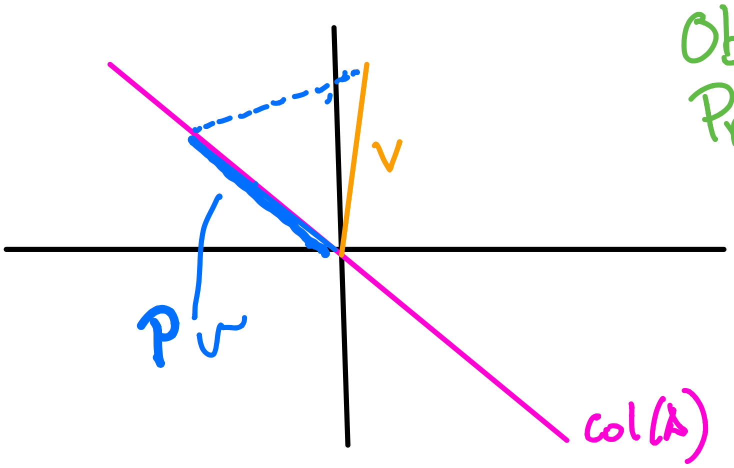
$$P^2 = P \quad (\text{idempotent matrix.})$$

Projectors can be orthogonal or non-orthogonal.



ORTHOGONAL PROJECTION

" Pv is a projection of v onto col(A)"

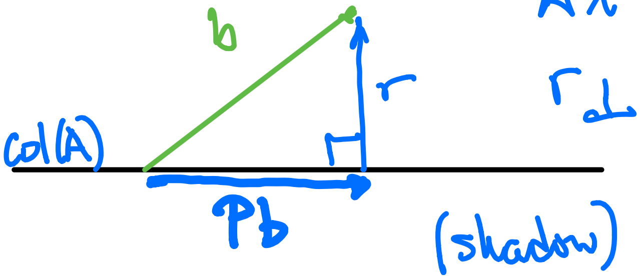


Oblique Projection

An application (more later) is LEAST SQUARES:

$$Ax = b$$

b may not be in col(A) and thus we'd settle for a "projected" solution



(shadow)

$Ax - b = r$

$r \perp$ to col(A)

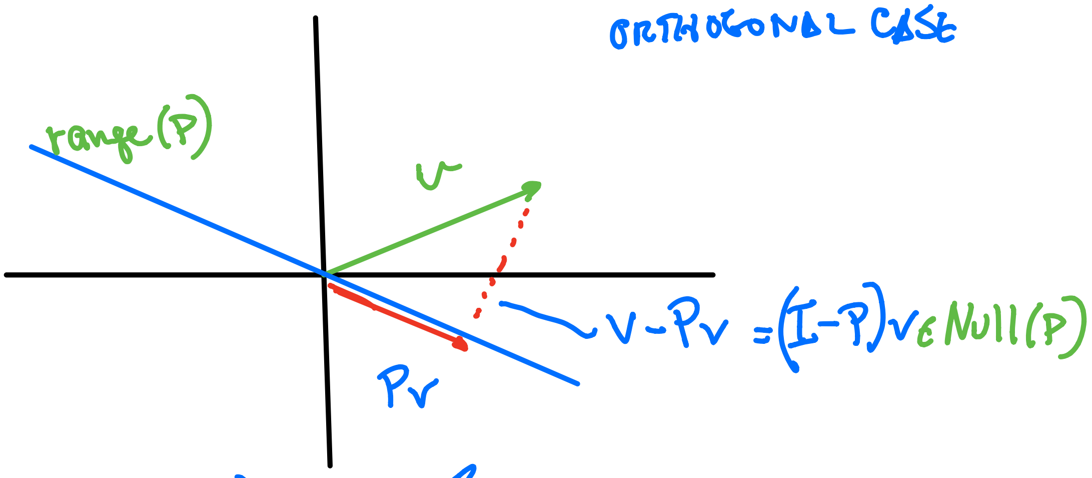
If $v \in$ col(P), i.e. v is in the range of P then v lays on its own "shadow"

$$v = Px \quad \text{for some } x$$

$$Pv = PPx = P^2x = Px = v$$

since $P^2 = P$

ORTHOGONAL CASE



$$P(v-Pv)=Pv-P^2v=0$$

$v-Pv \in null(P)$

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The Algebra of Projectors:

if P is a projector **I-P** is called the

COMPLEMENTARY PROJECTOR

I-P projects onto the null space of P:

if $Pv=0 \Rightarrow (I-P)v=v$

and $(I-P)v=v-Pv$, for any v

$\therefore$ the range of $I-P = null(P)$ //

Rmk: $P=I-(I-P) \Rightarrow null(I-P)=range(P)$

$\text{null}(I-P) \cap \text{null}(P) = \{0\}$ ∴ Any vector in both the subspaces satisfies $v = v - Pv = (I-P)v = 0$

$//$

A Projector in $\mathbb{C}^m$ separates it into 2 spaces: $\mathbb{C}^m = S_1 \oplus S_2$, where $S_1 \in \mathbb{C}^n$, $S_2 \in \mathbb{C}^{(n+1,m)}$

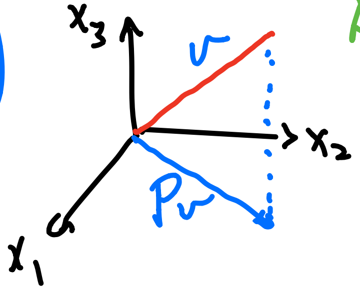
if $v \in \mathbb{C}^m$ then $Pv = \text{Range}(P)$

$(I-P)v = \text{null}(P)$

$$\left[\begin{array}{c} S_2 \oplus S_1 = \mathbb{C}^m \\ \text{null}(P) \oplus \text{range}(P) = \mathbb{C}^m \end{array}\right]$$

$//$

ex)



AN EXAMPLE OF AN ⊥ PROJECTOR:

$v \in \mathbb{R}^3$

$Pv \in \mathbb{R}^3$ (but lives in $\mathbb{R}^2$)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Px = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in S_1$$

$$(I-P)x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \in S_2 \qquad S_1 \oplus S_2 = \mathbb{R}^3$$

ORTHOGONAL PROJECTORS

$$\begin{cases} P^2 = P \text{ and} \\ P^* = P \quad \text{Hermitian} \end{cases}$$

ex) $A \in \mathbb{C}^{m\times m}$ & s'pse A has m e'vectors $\{v_j\}_{j=1}^{m}$. $\{v_j\}$ a basis of $\mathbb{C}^m$, s'pse v_i's $\perp$

Take $x \in \mathbb{C}^m$, we can write as linear superposition

$$x = a_1 v_1 + a_2 v_2 \cdots a_m v_m$$

To find $a_i = \dfrac{v_i^* x}{\|v_i\|^2}$. Hence

think of $P = \dfrac{v_i v_i^*}{\|v_i\|^2}$ as a rank 1 matrix that projects x onto v_i subspace. //

Th'm: An orthogonal projector P is any projector that is Hermitian ($P^* = P$)

Pf: let $x \in V$ and $y \in V$

Suppose $Px \in S_1$

$(I-P)y \in S_2$

if $P^* = P$, compute the inner product

$$(Px)^*(I-P)y = x^*P^*(I-P)y = x^*P^*y - x^*P^*Py$$

$$x^*P^*y - x^*P^{*2}y = 0$$

$\therefore$ P is $\perp$ projector //

Thm: P is $\perp$ projector iff $P = P^*$

S'pse $\dim(S_1) = n$ and $S_1 \perp S_2$

S'pse $x \in \mathbb{C}^m$

Find $\{q_1, q_2 \dots, q_n\}$ $\perp_1$ (orthonormal) basis for $\mathbb{C}^n$

let $y \in S_2$ in space $\{q_{n+1}, q_{n+2} \dots, q_m\}$

Want a P such that $Pq_j = q_j$ if $1 \leq j \leq n$

$Pq_j = 0 \quad \text{if } n+1 \leq j \leq m$

let $Q = \underbrace{\begin{bmatrix} | & | & & | & & & \\ q_1 & q_2 & \cdots & q_n & 0 & 0 \cdots & 0 \\ | & | & & | & & & \end{bmatrix}}_{m}$

so $PQ = Q$

and $Q^*PQ = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 \ddots & \\ & & & & & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\ \ddots\\1\end{matrix}}\right\}n \\ \vphantom{\begin{matrix}0\\0\end{matrix}} \end{matrix} \Bigg] m \equiv \Sigma$

$$P = Q\Sigma Q^*$$

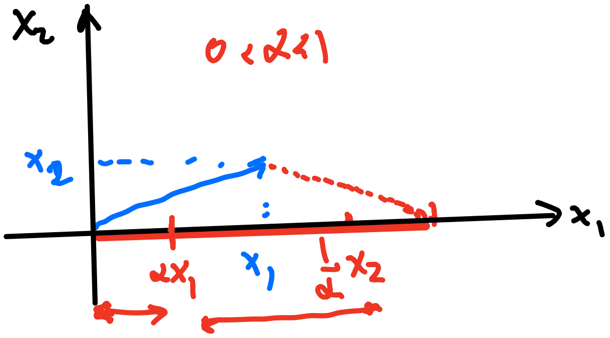
$$P^* = (Q\Sigma Q^*)^* = Q\Sigma^* Q^* = Q\Sigma Q^*$$

$$\therefore \quad P^* = P \qquad //$$

Example of an Oblique Projector: with $\alpha \neq 0$,

$P = \frac{1}{\alpha}\begin{pmatrix} \alpha^2 & 1 \\ 0 & 0 \end{pmatrix} = P^2$ & P not Hermitian

$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \frac{1}{\alpha}x_2 \\ 0 \end{pmatrix}$$



S'pse P is orthogonal

$P = Q \Sigma Q^*$ FULL REPRESENTATION

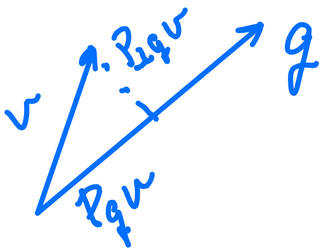
$P = \hat{Q} \hat{Q}^*$ REDUCED REPRESENTATION

REDUCED ORTH COMPLEMENT: $I - \hat{Q} \hat{Q}^*$

Special Case: Rank 1 Projector

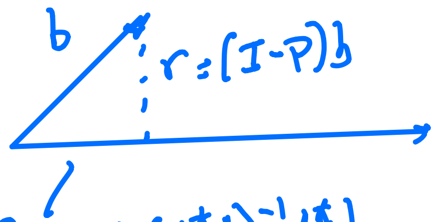
$P_q = q q^*$ projector in the direction q

$P_{\perp q} = I - q q^*$



Arbitrary case $P_{\hat{a}} = \frac{a a^*}{\|a\|^2}$ $P_{\hat{a} \perp} = I - \frac{a a^*}{\|a\|^2}$

So: $Ax - b = r$



$$Pb = A(A^*A)^{-1}A^*b$$

$$b = r + Pb$$

QR Decomposition (Lecture 7)

(G-S algorithm)

$A \in \mathbb{C}^{m\times n}$ $m \geq n$

furthermore, assume A has full rank $r = n$:

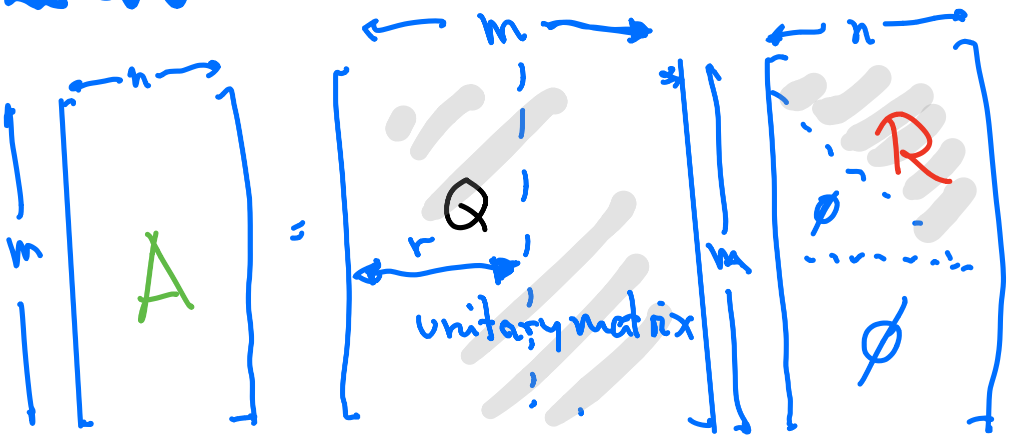
$$A = \hat{Q}\hat{R}$$

$$m\updownarrow \overbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \\ & & & \end{bmatrix}}^{n} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \\ & & & \end{bmatrix} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & 0 & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}}_{\text{upper triangular}}$$

A $=$ $\hat{Q} \in \mathbb{C}^{m\times n}$ $\hat{R} \in \mathbb{C}^{n\times n}$

$$\left\{\begin{aligned} a_1 &= r_{11} q_1 \\ a_2 &= r_{12} q_1 + r_{22} q_2 \\ &\vdots \\ a_n &= r_{1n} q_1 + r_{2n} q_2 + \cdots + r_{nn} q_n \end{aligned}\right.$$

FULL QR:



COMPUTING QR=A

Recall G-S Given $a_1, a_2 \ldots, a_n$ (s'pse a_i L.I)

$$v_j = a_j - (q_1^* a_j) q_1 - (q_2^* a_j) q_2 \cdots (q_{j-1}^* a_j) q_{j-1}$$

$$q_j \in \langle a_1 \ldots, a_j \rangle$$

$v_j = 0$ if $v_j \in \langle a_1, a_2 .., a_j \rangle$, otherwise $v_j \neq 0$.

Here, $q_j = \frac{v_j}{\|v_j\|}$. Hence

$$q_1 = \frac{a_1}{r_{11}} , \quad q_2 = \frac{a_2 - r_{12} q_1}{r_{22}}$$

$$\vdots$$

$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in} q_i}{r_{nn}}$$

$$\Rightarrow r_{ij} = q_i^* q_j \qquad (i \neq j)$$

$$|r_{jj}| = \| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \|_2$$

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