SVD:

$$A = U\Sigma V^* \qquad A \in \mathbb{C}^{m\times n}$$

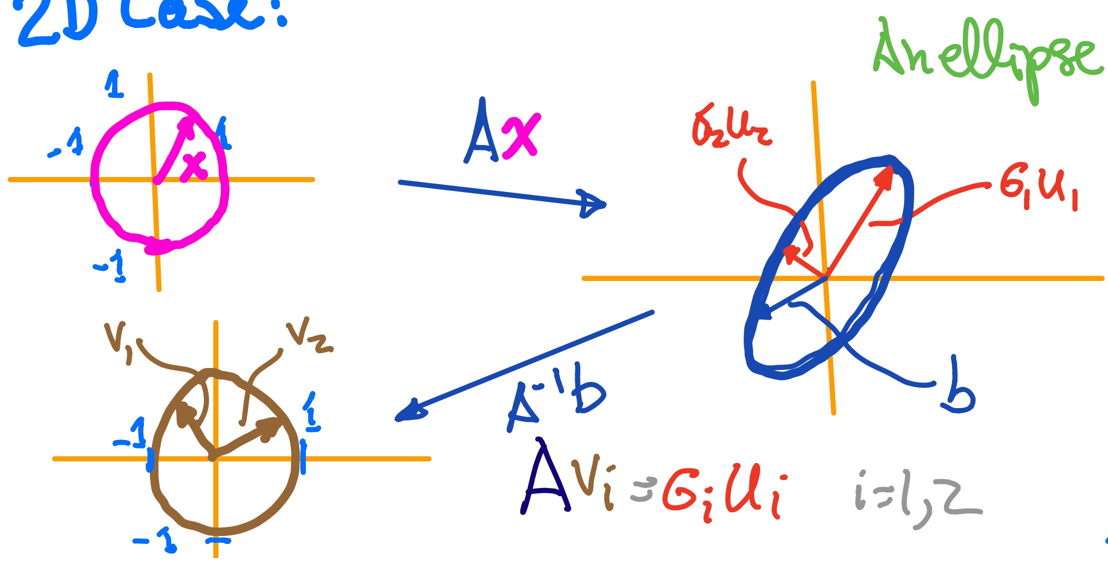
$$\left(U\Sigma V^T \quad \text{if } A \in \mathbb{R}^{m\times n} \right)$$

$$\begin{cases} AA^* U = U\Sigma\Sigma^* & \text{(Left)} \\ A^*A V = V\Sigma^*\Sigma & \text{(Right)} \end{cases}$$

both are e'value problems with e'values $\sigma_1^2, \sigma_2^2, \dots, \sigma_r^2$ //

Observation pick $\{x : \|x\|_2 = 1, x \in \mathbb{R}^n\}$ & $A \in \mathbb{R}^{m\times n}$ then $Ax = b \in \mathbb{R}^m$ is a vector set that traces out a hyperellipse.

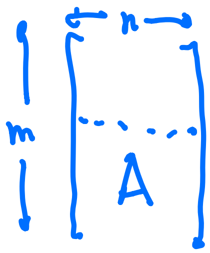
2D Case:



$A v_i = \sigma_i u_i \quad i = 1, 2$ //

If A has rank $r \Rightarrow r$ σ_i's are nonzero

σ_i are called the singular values of A



at most n will be nonzero

The Hyperellipse: Consider $r = n$

& the semiaxis have lengths $\sigma_1 \geq \sigma_2 \cdots \sigma_n > 0$

n LEFT singular vectors of A: $\{u_1, u_2, \dots, u_n\}$

u_i's are unit, orthogonal. These are the semiaxes directions of hyperellipse.

n RIGHT singular vectors $\{v_1, v_2 \dots, v_n\}$

$V = \{v_1, v_2 \dots, v_n\}$ if $S \in V$ then

$T = AS \in \{u_1, u_2 \dots, u_n\}$

//

REDUCED SVD:

$$Av_j = \sigma_j u_j \qquad 1 \leq j \leq n$$

$$\underset{\displaystyle A}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{m\times n} \underset{\displaystyle V}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{n\times n} = \underset{\displaystyle \hat{U}}{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}}_{m\times n} \underset{\displaystyle \hat{\Sigma}}{\begin{bmatrix} \sigma_1 & & \phi \\ & \sigma_2 \ddots & \\ \phi & & \sigma_n \end{bmatrix}}_{n\times n}$$

$$AV = \hat{U}\hat{\Sigma}$$

Reduced SVD: $A = \hat{U}\hat{\Sigma}V^*$ //

FULL SVD

replace $\begin{cases} \hat{U} \to U & (m\times m) \\ \hat{\Sigma} \to \Sigma & (m\times n) \end{cases}$

fullrank:

$$\begin{array}{c} \\ m \end{array} \overset{n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} = \overset{m}{\begin{bmatrix} & \vdots & \\ \hat{U} & \vdots & \\ & \vdots & \end{bmatrix}}_{m} \overset{n}{\begin{bmatrix} \sigma_1 & & \phi \\ & \sigma_2 \ddots & \\ \phi & & \sigma_n \\ \cdots & \cdots & \cdots \\ & \phi & \end{bmatrix}} \overset{n}{\begin{bmatrix} & & \\ & V^* & \\ & & \end{bmatrix}}_{n} \quad \hat{\Sigma}$$

$$A = U \quad \Sigma \quad V^*$$

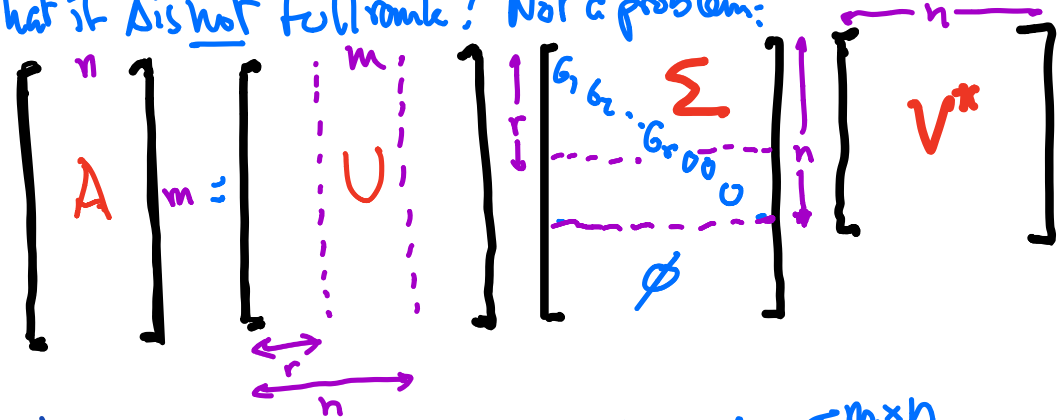
$A \in \mathbb{C}^{m\times n}$

$U \in \mathbb{C}^{m\times m}$ unitary spans $\mathbb{C}^m$

$V \in \mathbb{C}^{n\times n}$ " " $\mathbb{C}^n$

$\Sigma \in \mathbb{C}^{m\times n}$ singular values along diagonal //

What if A is not full rank? Not a problem:



EXISTENCE & UNIQUENESS: Every $A \in \mathbb{C}^{m \times n}$ has an SVD. Further, the $\{\sigma_j\}$ are **uniquely** determined. If $A \in \mathbb{C}^{n \times n}$ & $\{\sigma_j\}$ are **distinct** $\Rightarrow$ $\{u_i\}$ & $\{v_i\}$ are **unique**, to within multiplicative constants //

LECTURE 5 (MATRIX FACTS VIA SVD):

$b = Ax \qquad A \in \mathbb{C}^{m \times n}, \; b \in \mathbb{C}^m$ **expand**

$$\begin{cases} b = \sum_{i=1}^{m} \alpha_i u_i & \{u_i\} \text{ span } \mathbb{C}^m \\ x = \sum_{i=1}^{n} \beta_i v_i & \{v_i\} \text{ span } \mathbb{C}^n \end{cases}$$

or

$$b = U\alpha \quad \text{and} \quad x = V\beta$$

α, β are the vectors of coefficients $\{\alpha_i\}, \{\beta_i\}$

Since U & V are unitary

(if Q is unitary $Q^*Q = I$ i.e. $Q^* = Q^{-1}$)

$U^*b = \alpha$ $\qquad$ $V^*x = \beta$

so $b = Ax$

$$U^*b = U^*Ax = U^*(U\Sigma V^*)\underbrace{x}_{V\beta}$$

$$\iff \alpha = \Sigma\beta$$

if you choose the right bases for $\mathbb{C}^m$ and $\mathbb{C}^n$ then SVD shows that the linear transformation can be written as a diagonal map! //

Relation between SVD and DIAGONALIZATION:

if A is diagonalizable

$$A = X\Lambda X^{-1}$$

X columns are e'vectors of A

Λ matrix with e'values as diagonal entries

if $b = Ax$ & $b = X\beta$ $\quad x = X\alpha$

then $b = Ax \iff X\beta = Ax = (X\Lambda X^{-1})X\alpha$

or $\beta = \Lambda\alpha$ //

SVD vs Diagonalization (EV)

SVD uses 2 different bases

SVD uses orthonormal vectors
EV uses L.I vector

Not all A have a diagonalization, even if A is square. However, all A have an SVD.

MATRIX PROPERTIES:

let $A \in \mathbb{C}^{m \times n}$

$p \equiv \min(m,n)$

$r \leq p$ nonzero singular σ_i

$\langle x, y, \ldots \rangle \equiv$ space spanned by $x, y \ldots$

Thm: $\operatorname{rank}(A) = r$

pf: $\operatorname{rank}(A) = \operatorname{rank}(U \Sigma V^*) = \operatorname{rank}(\Sigma) = r$

Thm $\operatorname{col}(A) = \operatorname{range}(A) = \langle u_1, u_2, \ldots, u_r \rangle$

$\operatorname{null}(A) = \langle v_{r+1}, v_{r+2}, \ldots, v_n \rangle$

Thm: $\|A\|_2 = \sigma_1$

$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$

Th'm: $\sigma_i > 0$ are the square roots of $\lambda_i > 0$ e'values of A^*A and AA^*: recall that

$$\begin{cases} AA^*U = U\Sigma\Sigma^* \\ A^*AV = V\Sigma^*\Sigma \end{cases}$$

//

$$\boxed{\begin{array}{l}\text{Th'm:}\\ \text{If } A = A^* \text{ (Hermitian)} \Rightarrow A = \underset{U}{Q}\,\underset{\Sigma}{\Lambda}\,\underset{V^*}{Q^*} \\ Q \text{ is complete \& the entries of } \Lambda \text{ are real //}\end{array}}$$

Thm: $|\det(A)| = \prod_{i=1}^{m} \sigma_i$

$$|\det(U\Sigma V^*)| = |\det U \det\Sigma \det V^*|$$
$$= |(\pm 1)\det\Sigma(\pm 1)| = |\det\Sigma|.$$

Since Σ is diagonal then

$\det\Sigma = \prod$ diagonal elements of Σ //

LOW RANK APPROXIMATIONS

Thm: A is the sum of rank 1 matrices. Follows from

$$A = \sum_{j=1}^{n} \sigma_j u_j v_j^* = U\Sigma V^*.$$

A very useful quality of rank 1 representations of matrices is that the k^{th} partial sum has the highest possible 2-norm or F-norm:

Thm: For any $0 \le k \le r$ let

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^* .$$

If $k = p = \min(m,n) \Rightarrow \sigma_{k+1} = 0$.

$$\|A - A_k\|_2 = \inf_{\substack{B \in \mathbb{C}^{m \times n} \\ \operatorname{rank}(B) \le k}} \|A - B\|_2 = \sigma_{k+1}$$

//

Thm: For any $0 \le k \le r$ the matrix A_k also satisfies

$$\|A - A_k\|_F = \inf_{\substack{B \in \mathbb{C}^{m \times n} \\ \operatorname{rank}(B) \le k}} \|A - B\|_F = \sqrt{\sigma_{k+1}^2 + \dots + \sigma_r^2}$$

//