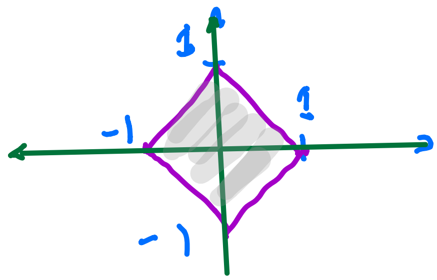
# p-norms Vector case:

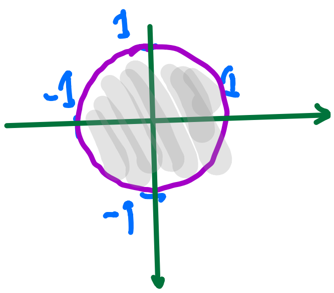
Take $x \in \mathbb{R}^2$ $\quad x = x_1\hat{e}_1 + x_2\hat{e}_2$

Find $\|x\| \le 1$:

1-norm $\quad \|x\|_1 = |x_1| + |x_2|$

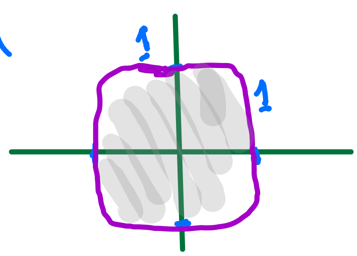



2-norm



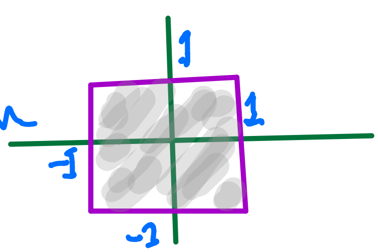

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$$

p-norm




$$\|x\|_p = \left(|x_1|^p + |x_2|^p\right)^{1/p}$$

∞-norm




$$\|x\|_\infty = \max_{\substack{1 \le i \le m \\ \downarrow \\ 2}} \left(|x_1|, |x_2|\right)$$

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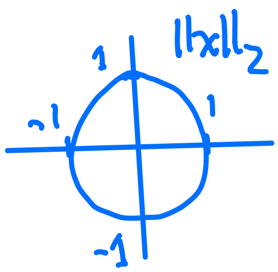
# The Weighted p-norm:

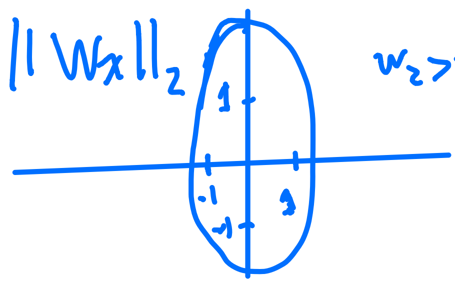
$$\|x\|_W = \|Wx\|$$

e.g. $W$ might be a diagonal matrix with $w_i > 0$,

$$\|x\|_W = \left( \sum |w_i x_i|^2 \right)^{1/2}$$

In 2D: $W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ $\quad w_i > 0$




$w_2 > w_1 > 1$

## Useful facts about $L_1, L_2, L_\infty$

$L_p$ norm of $\begin{cases} f(x) & a \le x \le b \\ f(x) & \text{real} \end{cases}$

a constant

$$\|f\|_1 = \int_a^b |f(x)|\,dx \le \max_{a \le x \le b} |f(x)| \int_a^b dx = (b-a)\,\|f\|_\infty$$

$$\therefore \|f\|_1 \leq c_1 \|f\|_\infty$$

$$\|f\|_2^2 = \left[ \int_a^b |f(x)|^2 \, dx \right] \leq \max |f|^2 \int_a^b dx$$

$$\|f\|_2 \leq c_2 \|f\|_\infty$$

Rmk: Suppose $f(x) = p(x) - g(x)$ $\therefore \|f\|$ measures how close $p$ is to $g$. We recognize $\|f\|_\infty$ as the "uniform convergence" norm.

Convergence in the $\infty$-norm is strongest.

**Norms of Matrices:** Let $A \in \mathbb{C}^{m \times n}$ $Ax = b$

$x \in \mathbb{C}^n$ $b \in \mathbb{C}^m$

Recall $\|A\|_{(m,n)} = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_{(n)} = 1}} \|Ax\|_{(m)}$

is the norm.

How do we compute this?

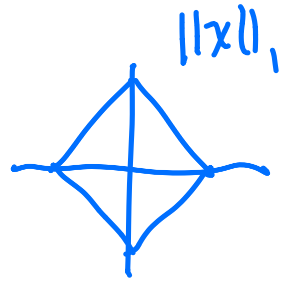
Let's look at an easy case:

Take Diagonal Matrix $D \in \mathbb{R}^{2\times 2}$ first:
(for some intuition)

$\|D\|_1$

Take $\left\|\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\| = |d_1|$

$\left\|\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\| = |d_2|$

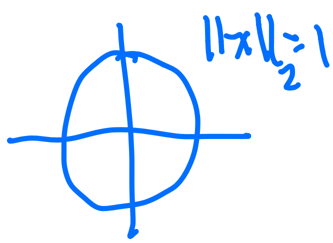


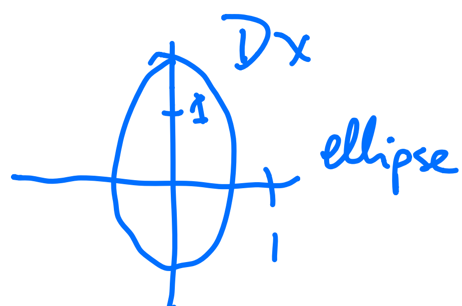

$\|D\|_1 = \max[|d_1|\ |d_2|]$. Generalizing:

Suppose $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$:

$$\|D\|_1 = \max_{1 \le i \le m} |d_i|$$

$\|D\|_2$

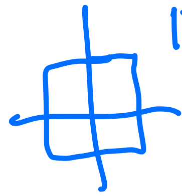


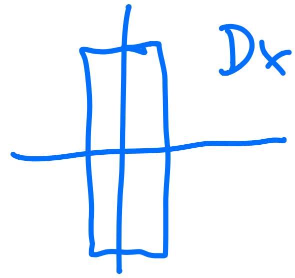

$$\|D\|_2 = \max_{1 \le i \le m} |d_i|$$

(Hyperellipse)

$\|D\|_\infty$



$\|x\|_\infty = 1$

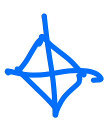



$$\|D\|_\infty = \max_{1 \le i \le n} |d_i|$$

# General Case:

- **1-norm:** "maximum column sum" of $A$

$$A = \begin{array}{c} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ & | & & | \end{bmatrix} & m \end{array}$$

in 2D  $\|x\|_1 = 1$

$$\|Ax\|_1 = \|a_1x_1 + a_2x_2 \cdots a_nx_n\|_1$$

$$\le \sum_{j=1}^{n} |x_j| \, \|a_j\|_1$$

$$\|A\|_1 = \max_{1 \le j \le n} \|a_j\|_1 = \max \left[ \sum_{i=1}^{m} |a_{ij}| \right]$$

- $\infty$-norm "maximum row sum of A"

$$\|A\|_\infty = \max_{1\le i\le m}\left[\sum_{j=1}^{n}|a_{ij}|\right]$$

The Frobenius Norm

- $$\|A\|_F = \left[\sum_{j=1}^{n}\sum_{i=1}^{m}|a_{ij}|^2\right]^{1/2}$$

Also can be found as:

$$\|A\|_F = \sqrt{tr(A^*A)} = \sqrt{tr(A^*A)}$$

Rmk: $tr(B) = \sum_{i=1}^{n} b_{ii}$ for $B\in\mathbb{R}^{n\times n}$ //

- 2-norm

$$\|A\|_2 = \left[\,|\text{largest eigenvalue}| \text{ of } A^*A\right]^{1/2}$$

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These definitions for matrix norms are called compatible because:

$$\|Ax\|_1 \le \|A\|_1 \|x\|_1$$

$$\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$$

$$\|Ax\|_2 \le \|A\|_F \|x\|_2$$

## Hölder's Inequality

Let $p, q$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$

$1 \le p, q \le \infty$

Hölder's Inequality

$$|x^* y| \le \|x\|_p \|y\|_q$$

Special Case: Cauchy Schwarz $p = q = 2$

$$|x^* y| \le \|x\|_2 \|y\|_2$$

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Bounding $\|AB\|$

$B \in \mathbb{C}^{m\times n}$ $\quad A \in \mathbb{C}^{\ell\times m} \Rightarrow AB \in \mathbb{C}^{\ell\times n}$

Assume $x \in \mathbb{C}^n$

$$\|ABx\|_{(\ell)} \le \|A\|_{(\ell,m)} \|Bx\|_{(m)}$$

$$\le \|A\|_{(\ell,m)} \|B\|_{(m,n)} \|x\|_{(n)}$$

$$\therefore \|AB\|_{(\ell,m)} \le \|A\|_{(\ell,n)} \|B\|_{(m,n)}$$

ex) $\|AB\|_F^2 \le \|A\|_F^2 \|B\|_F^2$

**Special Case:** the matrix product involving a unitary transformation:

<u>Th'm</u> For any $A \in \mathbb{C}^{m\times n}$ and $Q^{m\times m}$ unitary

$$\|QA\|_2 = \|A\|_2$$

$$\|QA\|_F = \|A\|_F$$

# SVD Singular Value Decomposition

Closely related to the e'value/e'vector factorization of a symmetric matrix for $A \in \mathbb{R}^{m \times m}$

$$A = QDQ^T$$

For most matrices this factorization is not possible (can't find a complete set of e'vectors)

Want a decomposition for general rectangular matrix in which the left matrix and right matrix are orthogonal but they are no longer always transposes (adjoint) of each other. We want the diagonal matrix $D$ to have non-negative entries.

Let $D = \Sigma$ with positive entries

$\sigma_1, \sigma_2, \ldots, \sigma_r$ singular values

rank $\Sigma = r$

The key is to use a square matrix, so

We'll use $A^*A$ or $AA^*$

<u>SVD</u>: For any $A \in \mathbb{C}^{m\times n}$

$$(\ddagger) \quad A = Q_1 \Sigma Q_2^* \text{ is an SVD}$$

with $Q_1$ & $Q_2$ (orthogonal) unitary matrices

The cols of $Q_1$ $(m\times m)$ are e'vectors of $AA^*$
The " " $Q_2$ $(n\times n)$ " " " $A^*A$
The $r$ singular values of $\Sigma$ are diagonal entries. The matrix $\Sigma$ $(m\times n)$ will have diagonal entries consisting of the square roots of the nonzero e'values of $A^*A$ or $AA^*$, and the rest are zero in the diagonal:

<u>Remark</u>: First $r$ cols of $Q_1$ = col$(A)$
last $m-r$ cols of $Q$ = null$(A^*)$
first $r$ cols of $Q_2$ = row$(A)$
last $n-r$ cols of $Q_2$ = null$(A)$ //

Note that: from $(\ddagger)$ we see that

$$AQ_2 = Q_1 \Sigma$$

The connection between SVD and $A^*A$ and $AA^*$:

$$A = Q_1 \Sigma Q_2^*$$

$$AA^* = (Q_1 \Sigma Q_2^*)(Q_1 \Sigma Q_2^*)^* = Q_1 \Sigma \Sigma^* Q_1^*$$

$$A^*A = Q_2 \Sigma^* \Sigma Q_2$$

Note $\Sigma = \Sigma^*$

So

$$AA^* Q_1 = Q_1 \Sigma\Sigma \quad (m\times m) \text{ e'value problem}$$

$$AA^* Q_2 = Q_2 \Sigma\Sigma \quad (n\times n) \text{ e'value problem}$$

ex) $A$ is diag, $A \in \mathbb{R}^{3\times 2}$, Show that:

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q_2^*}$$

ex) $A$ is $\mathbb{R}^{3\times 1}$, show that

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \underbrace{\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{Q_2^*}$$