THE INNER PRODUCT: for $x, y \in \mathbb{C}^m$

$$x^* y = \sum_{i=1}^{m} \bar{x}_i y_i$$

ex) Euclidean length: $x \in \mathbb{C}^m$

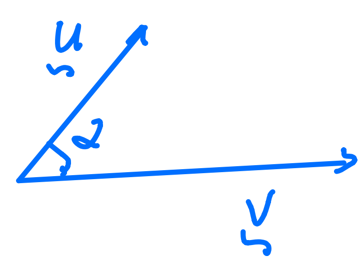
$$\|x\| = \sqrt{x^* x} = \sqrt{\sum_{j=1}^{m} |x_j|^2}$$

Note that: $x_j = \alpha_j + i\beta_j$ then

$$|x_j|^2 = (\alpha_j + i\beta_j)(\alpha_j - i\beta_j) = \alpha_j^2 + \beta_j^2$$

The Dot Product

$$\underset{\sim}{v} \cdot \underset{\sim}{u} = \underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}| \cos\alpha$$



if $\underset{\sim}{u}, \underset{\sim}{v}$ are real then $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u}^T \underset{\sim}{v}$

Some Important Vector/matrix Properties:

Vectors:

$$(\alpha x)^*(\beta y) = \bar{\alpha}\beta x^* y$$

$$\alpha, \beta \in \mathbb{C}$$

$$(x y)^* = y^* x^*$$

Matrices, provided A & B are compatible

$$(AB)^* = B^* A^*$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$A^{-*} \equiv (A^{-1})^* = (A^*)^{-1}$$

ORTHOGONAL VECTORS

2 vectors are orthogonal $\perp$ if $\underline{x}^* \underline{y} = 0$

if $\underline{x} \neq \emptyset$ $\underline{y} \neq 0$

$\underline{x}^* \underline{y} = |\underline{x}||\underline{y}| \cos\alpha \Rightarrow \alpha = 90°$

2 sets X & Y are said to be $\perp$ if all elements of X are $\perp$ to all elements of Y.

ORTHONORMAL VECTORS: $\perp$ have norm $= 1$

<u>Thm 2.1</u> Orthogonal vectors are LI (linearly independent)

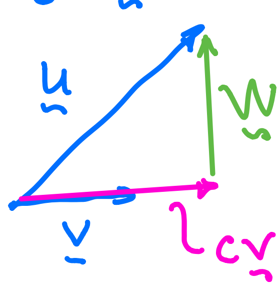
<u>Thm 2.2</u> If an orthogonal set $S \subseteq \mathbb{C}^m$ contains m vectors $\Rightarrow$ the set S spans $\mathbb{C}^m$

$$S = \{s_1, s_2, \dots, s_m\}$$

vector space $\mathbb{C}^m = \{s_1\} \oplus \{s_2\} \oplus \dots \oplus \{s_m\}$

is a DIRECT SUM OF SUBSPACES $\{s_i\}_{i=1}^{m}$

<u>**Projection**</u>: consider a simple example: the projection of $\underset{\sim}{u}$ onto the vector $\underset{\sim}{v}$:



$$\underset{\sim}{u} = c\underset{\sim}{v} + \underset{\sim}{w}, \quad c = \frac{u^* v}{\|v\|^2}$$

$\underset{\sim}{u}$ has been written in terms of an orthogonal basis

where $\underset{\sim}{w} = \underset{\sim}{u} - \frac{u^* v}{\|v\|^2} \underset{\sim}{v}$.

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3D Case

Suppose we know 3 LI vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$

Want to produce 3 vectors $\underline{q}_1, \underline{q}_2, \underline{q}_3$ $\perp$ and normalized (orthonormal)

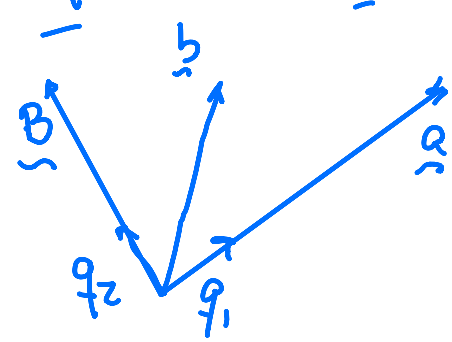
① Pick $\underline{q}_1 \parallel \underset{\sim}{a}$ so $\underset{\sim}{q}_1 = \dfrac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|}$

② Find $\underline{q}_2$: if $\underset{\sim}{b}$ has components in the q_1 direction, these have to be subtracted

$$\underset{\sim}{B} = \underset{\sim}{b} - (\underline{q}_1^* \underset{\sim}{b})\underline{q}_1$$

$$\underline{q}_2 = \frac{\underset{\sim}{B}}{\|\underset{\sim}{B}\|}$$ note that

$\underline{q}_2 \perp \underline{q}_1$ & $\|\underline{q}_1\| = \|\underline{q}_2\| = 1$.



So q_3 cannot be in the plane spanned by q_1 & q_2. So c should not have any components in Span$\{q_1, q_2\}$

$$C = c - (q_1^* c)\, q_1 - (q_2^* c)\, q_2$$

$$q_3 = \frac{C}{\|C\|}$$

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Generally: Suppose $v \in \mathbb{C}^m$ $m \geq n$:

S'pse $\{q_1, q_2, \ldots, q_n\}$ are $\perp$, normal. Let v be an arbitrary vector. Then

$q_j (q_j^* v)$ is a scalar projection of v onto q_j. So

$$v - (q_1^* v) q_1 - (q_2^* v) \cdots (q_n^* v) q_n \equiv r$$

$r \perp \{q_1, q_2 \ldots q_n\}$, since

$$q_i^* r = q_i^* v - q_i^* (q_1^* v) q_1 - \ldots q_i^* (q_n^* v) q_n$$

$$\underset{\sim}{q}_i^* \underset{\sim}{r} = \underset{\sim}{q}_i^* \underset{\sim}{v} - (\underset{\sim}{q}_i^* \underset{\sim}{v})(\underset{\sim}{q}_i^* \underset{\sim}{q}_i)^{=1} = 0$$

Hence the generalization of ($\cancel{\star}$) is

$$\underset{\sim}{v} = \underset{\sim}{r} + \sum_{i=1}^{n} (\underset{\sim}{q}_i^* \underset{\sim}{v}) \underset{\sim}{q}_i$$

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UNITARY MATRICES

S'pse matrix $Q \in \mathbb{C}^{m \times m}$ is Unitary

(if Q is real we say Q is orthonormal)

Then $Q^* = Q^{-1}$ $\qquad$ $Q^*Q = I$

$$\underset{\sim}{q}_i^* \underset{\sim}{q}_j = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

The cols of Q form an orthonormal basis for $\mathbb{C}^m$.

Multiplication by Unitary Matrix

Recall that $b = Ax$ is interpreted as x, the set of coeff's that uniquely represent $\underset{\sim}{b}$ in terms of cols of A

Take $A = Q$. Then

$$b = Qx$$

is the representation of b as a direct sum of subspaces of $\mathbb{C}^m$:

Q^*b finds the x coefficients.

Multiply by Q^* $\longrightarrow$

Q^*b coeffs of expansion of b in terms of $\{q_1, q_2, \dots, q_m\}$

b: coeffs of expansion of b in the $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m\}$

$\longleftarrow$ Multiply Q

Property of Q: preserves the inner product:

$$(Qx)^*(Qy) = x^*Q^*Q\,y = x^*y$$

(invariance of angles)

Also

$$\|Qx\| = \|x\|$$

In real case, multiplication by orthogonal matrix Q corresponds to rigid rotation ($\det Q = 1$) or a reflection $\det Q = -1$ of vector space. //

Lecture 3 Vector Norms

L_p norm of a function $\underset{\sim}{f}(x)$

$$\left[\int_{\Omega} |\underset{\sim}{f}(x)|^p dx \right]^{1/p} \qquad p = 1, 2, \dots \infty$$

Ω is the domain of $f(x)$

Note: $\|f\|_\infty = \max_{x \in \Omega} |f(x)|$

The "discrete" version is the ℓ_p (p-norm)

$$\ell_p \quad \|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \qquad p = 1, 2, \dots \infty$$

for $x \in \mathbb{C}^m$

It gets confusing to have an Lp norm for the space of Lp functions so Trefethen calls the norms p-norms.

Vector Norm : is the non-negative function $\|\cdot\|$

$\|\cdot\| : \mathbb{C}^m \rightarrow \{\mathbb{R}^+, 0\}$. It assigns a length to each vector.

We use it to describe the size of vector quantities and/or distance between vector constructs.

Norm Properties

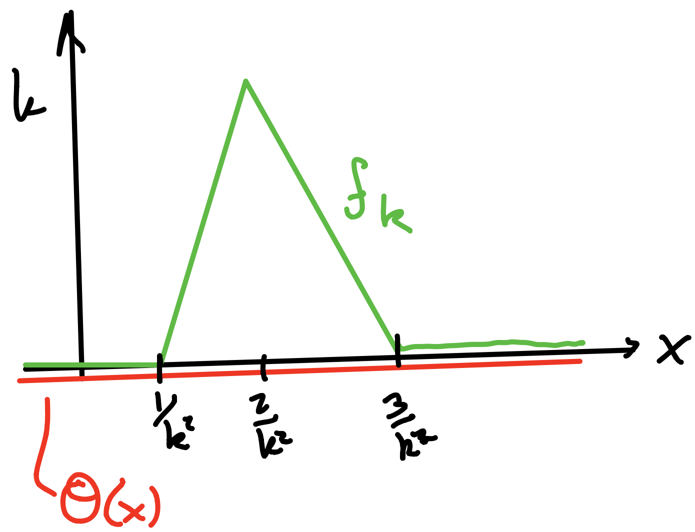
(1) $\|x\| \geq 0$ and $\|x\| = 0$ if $x = \emptyset$ (positivity)

(2) $\|x+y\| \leq \|x\| + \|y\|$ triangle inequality

(3) $\|\alpha x\| = |\alpha| \, \|x\|$ $\alpha \in \mathbb{C}$ scaling

ex) Let $\Theta(x) = 0$

let $$f_k = \begin{cases} k(k^2x-1) & \frac{1}{k^2} \leq x < \frac{2}{k^2} \\ -k(k^2x-1) & \frac{2}{k^2} \leq x < \frac{3}{k^2} \\ 0 & \text{otherwise} \end{cases}$$



if $\|\theta - f_k\|_1 = \frac{1}{k}$

if $\|\theta - f_k\|_2 = \frac{\sqrt{2}}{\sqrt{3}}$

if $\|\theta - f_k\|_\infty = k$

The L_1 norm decreases as $k \to \infty$

L_2 " constant

L_∞ " increase as $k \to \infty$

Matrices & Induced Norms

$A \in \mathbb{C}^{m \times n}$ $\quad x \in \mathbb{C}^n$

$Ax \in \mathbb{C}^m$

Want a notion of length or size for A:

$$\|Ax\|_{(m)} \leq C\,\|x\|_{(n)}$$

$$\|A\|_{(m,n)} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq \underset{\sim}{0}}} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

$$= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_{(n)}=1}} \|Ax\|_{(m)}$$

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