ex) The outerproduct : product of m-dimensional col vector u with n-dim row vector v.

Yields an $m \times n$ (rank 1) matrix

$$\begin{array}{cc} & \overset{\longleftarrow n \longrightarrow}{\begin{bmatrix} \quad v \quad \end{bmatrix}} \\ m \begin{bmatrix} \\ u \\ \\ \end{bmatrix} & \end{array} = \begin{array}{c} \longleftarrow n \longrightarrow \\ \begin{bmatrix} | & | & & \\ v_1 u & v_2 u & \cdots & v_n u \\ | & | & & \end{bmatrix} \end{array} m$$

The cols are all multiples of the same vector u.

(Also the rows are all multiples of the same vector v)

Hence rank 1.

ex) Form $b_j = \sum_{k=1}^{j} a_k \qquad 1 \le j \le n$

$$b_1 = a_1$$
$$b_2 = a_1 + a_2$$
$$\vdots \qquad \vdots$$
$$b_n = \sum_{k=1}^{n} a_k$$

$$\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ 0 & & 1 \end{bmatrix}$$

$$B \quad = \quad A \quad R$$

So $R = \{r_{ij}\}$ is an $n \times n$ matrix upper triangular

$$\begin{cases} r_{ij} = 1 & i \le j \qquad 1 \le i \le n \\ & \qquad\qquad 1 \le j \le n \\ r_{ij} = 0 & i > j \end{cases}$$

The problem $B = AR$ is a discretization of an indefinite integral (Volterra) operator

$$\underset{\sim}{V f}(t) \equiv \int_0^t \underset{\sim}{f}(s)\, ds$$

//

REVIEW OF 4 BASIC SUBSPACES OF A MATRIX TRANSFORMATION

$$A \in \mathbb{R}^{m \times n} = \begin{array}{c} \left[\begin{array}{cccc} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{array}\right] \end{array} \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array}$$

Row Space
Col Space
Null(A) Kernel
Null(A^T)

Recall how these are found:

ex) $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \in \mathbb{R}^{m \times n}$ $\quad m = 4$, $n = 5$

$A\underset{\sim}{x} = \underset{\sim}{b}$ $\qquad A: \mathbb{R}^n \to \mathbb{R}^m$

$\qquad\qquad\qquad \mathbb{R}^5 \to \mathbb{R}^4$

$\underset{\sim}{x} \in \mathbb{R}^n = \mathbb{R}^5$ $\qquad \underset{\sim}{b} \in \mathbb{R}^m = \mathbb{R}^4$

Find Col(A)   matlab: use rref command

Row echelon calculation yields

$$B = \begin{bmatrix} \cancel{1} & 4 & 0 & 2 & 0 \\ 0 & 0 & \cancel{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cancel{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

// are pivots

can show that columns

$$\begin{cases} b_2 = 4b_1 \\ b_4 = 2b_1 - b_3 \end{cases}$$

and $a_2 = 4a_1$ $a_4 = 2a_1 - a_3$

$$\text{Col}(A) = \begin{bmatrix} | & | & | \\ a_1, & a_3 & a_5 \\ | & | & | \end{bmatrix}$$

$\dim(\text{col}(A)) = 3 \equiv \text{rank} = r$

The $\dim(\text{Null}(A)) = n - r = 5 - 3 = 2$

Null(A) is where all vectors $\underset{\sim}{x}$ st

$A\underset{\sim}{x} = \emptyset$ $\underset{\sim}{x} \neq \emptyset$

To find Null(A): Solve $A\underset{\sim}{x} = 0$ or $B\underset{\sim}{x} = 0$

$$\underset{\sim}{x} = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_4$$

$$\text{So Null}(A) = \left[\begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

$\text{row}(A) = \text{col}(A^T)$ The hard way:

find A^T and then row echelon ops...

but to find row(A) we can just use the pivot rows of B:

$$\text{row}(A) = \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(\text{row}(A)) = 3 = \text{rank}$$

$\text{Null}(A^T)$ No short cut in calculation

Need to find y s.t. $A^T y = \emptyset$

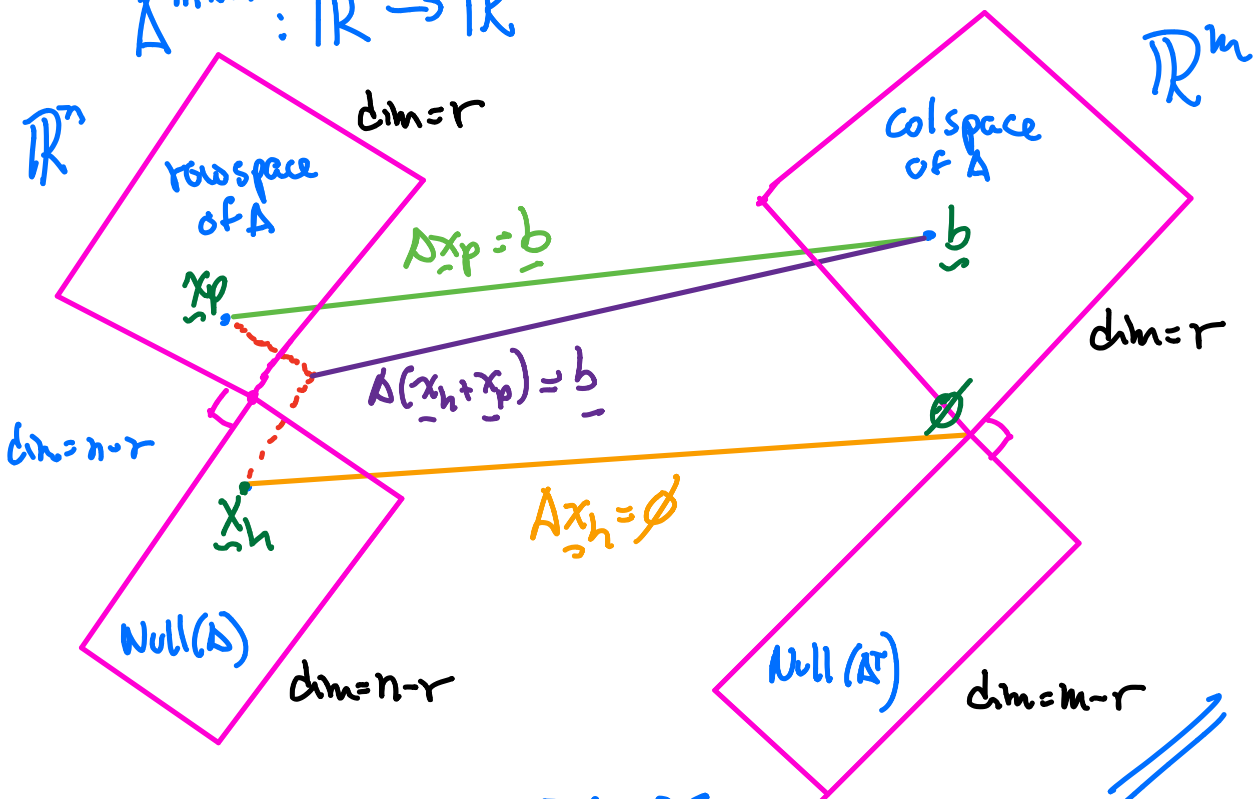
$$\dim(\text{Null}(A^T)) = 4 - 3 = 1$$

$$\begin{pmatrix} -1/3 \\ 1\,2/3 \\ 1 \\ -1\,1/3 \end{pmatrix} y_4 = y$$

GEOMETRIC INTERPRETATION OF A LINEAR

TRANSFORMATION: $A\underset{\sim}{x} = \underset{\sim}{b} = A(\underset{\sim}{x}_h + \underset{\sim}{x}_p) = \underset{\sim}{b}$ $\begin{cases} A\underset{\sim}{x}_h = 0 \\ A\underset{\sim}{x}_p = \underset{\sim}{b} \end{cases}$

$A^{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$



LINEAR INDEPENDENCE

The only solution to

$$A\underset{\sim}{x} = \emptyset \quad \text{is} \quad \underset{\sim}{x} = 0$$

if A is formed by linearly-independent column vectors.

Thm: For $A \in \mathbb{C}^{m \times m}$ the following are equivalent

(a) A has A^{-1}

(b) $\text{rank}(A) = m$

(c) $\text{range}(A) = \mathbb{C}^m$

(d) $\text{null}(A) = \{\emptyset\}$

(e) 0 is not an eigenvalue of A

(f) 0 is not a singular value of A

(e) $\det(A) \neq 0$

//

Matrix inverse times a vector

S'pse $b = Ax$ $\Rightarrow x = A^{-1} b$

Multiplication by A^{-1} is a "change of basis operation"

b: coeffs of expansion of b in $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$

$\xrightarrow[\text{multiply by}]{A^{-1}}$

$A^{-1}b$: coeffs of the expansion of b in $\{a_1, a_2 \dots a_m\}$

$\xleftarrow{A}$

$$\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \cdots + b_m \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

To find $\hat{e}_i = (A^{-1}A)_i$

$$I_m = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$I = AA^{-1} = A^{-1}A$$

//

Lecture 2 $\perp$ (orthogonal) matrices & vectors

<u>def</u>: Adjoint (Hermitian Conjugate)

for $A \in \mathbb{C}^{m\times n}$ $\quad (\cdot)^*$ is adjoint

say, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$ then $A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} & \bar{a}_{41} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} & \bar{a}_{42} \end{bmatrix}$

$\overline{(\cdot)}$ is complex conjugate

If $A \in \mathbb{R}^{m\times n}$ then $A^* = A^T$

<u>Hermitian Matrix</u> for $A \in \mathbb{C}^{m\times m} \Rightarrow$

if $A^* = A$ we say A is Hermitian.

If A is Hermitian it must also be square

If $A \in \mathbb{R}^{m\times m} \Rightarrow A = A^T$ (means symmetric)