Linear Superposition

We're familiar with the problem

(*) $Ax = b$,

where A and b are given, we then seek a solution x. We know x can be found if $b \in \text{span}(\text{columns of } A)$.

Let's think of (*) as

$$b = Ax$$

which says that b is a linear combination of the columns of A, x represents the coefficients or coordinates.

A is $m \times n$ matrix $A \in \mathbb{C}^{m \times n}$

rows cols

"complex"

$b \in \mathbb{C}^m \quad x \in \mathbb{C}^n$

$$A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}$$ the a_i are columns of A

(†) $b = x_1 a_1 + x_2 a_2 \cdots + x_n a_n$

makes b a linear combination of column vectors.

In index notation: $b_i = \sum_{j=1}^{n} a_{ij} x_j \quad i=1,2,\ldots,m.$

the i^{th} entry:

$$
m \uparrow\downarrow \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} = x_1 \begin{array}{c} j=1 \\ \begin{bmatrix} \vdots \\ a_{i1} \\ \vdots \end{bmatrix} \end{array} + x_2 \begin{array}{c} j=2 \\ \begin{bmatrix} \vdots \\ a_{i2} \\ \vdots \end{bmatrix} \end{array} + \cdots + x_n \begin{array}{c} j=n \\ \begin{bmatrix} \vdots \\ a_{in} \\ \vdots \end{bmatrix} \end{array} \uparrow\downarrow m
$$

(†) says b is a linear combination of column vectors

The map $x \to Ax$ is linear which means that it is closed under addition and scalar multiplication

$$A(x+y) = Ax + Ay$$

$$A(\alpha x) = \alpha Ax \qquad \alpha \in \mathbb{C}$$

Review: A collection $V = \{v_1 v_2 \ldots\}$ is a vector space if it is closed under addition and scalar multiplication.

$$\alpha v_i + \beta v_j = w \in V$$

$$v_i \,\&\, v_j \in V$$

$$\alpha, \beta \in \mathbb{C}$$

A vector space must have a "zero" element ϕ. It must have an "additive inverse"

$$v + w = \phi \quad \text{if } w = -v \in V$$

Linear Combination

$b = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is a linear combination, $\alpha_i \in \mathbb{C}$. $b \in \text{span}\{v_i\}_{i=1}^{n}$

A Transformation: $T: V \to W$ is a transformation from vector space V to W.

If T is a linear transformation

$$T(u+v) = T(u) + T(v)$$

$$T(\alpha v) = \alpha T v$$

$$\forall u \,\&\, v \in V \text{ and } \alpha \in \mathbb{C}^1$$

A linear transformation can be carried out via a matrix.

Linear Superposition: the effect of T on two vectors is the same as the sum of T applied to each vector:

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

$$\forall u, v \in V \quad \alpha, \beta \in \mathbb{C}^1$$

A nonlinear transformation example:

$T: x \to x^2$ $\quad y = x^2 = T(x)$

$T(\alpha x) = \alpha^2 x^2$

$T(\beta x) = \beta^2 x^2$

is $T(\alpha x + \beta x) = T(\alpha x) + T(\beta x)$?

if so, we would have a linear superposition

$T(\alpha x + \beta x) = \alpha^2 x^2 + 2\alpha\beta x^2 + \beta^2 x^2$

which is not $T(\alpha x) + T(\beta x) = \alpha^2 x^2 + \beta^2 x^2$ //

example of a linear transformation:

let $P_3 = \{\alpha_0 x^0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \mid \alpha_i \in \mathbb{R}\}$

is the set of all polynomials of degree at most 3.

linear superposition $\sum_{i=0}^{3} \alpha_i x^i + \sum_{i=0}^{3} \beta_i x^i = \sum_{i=0}^{3} (\alpha_i + \beta_i) x^i$

Also, note that for $r \in \mathbb{R}$

$$r \sum_{i=0}^{3} \alpha_i x^i = \sum_{i=0}^{3} r\alpha_i x^i$$

There's a sense in which we think of P_3 as being the "same" as $\mathbb{R}^4$: we say P_3 is isomorphic to $\mathbb{R}^4$ (same vector properties)

let $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = W$

where $e_0 = x^0$, $e_1 = x^1$, $e_2 = x^2$, $e_3 = x^3$

so
$$
\begin{array}{r}
1 - 2x + 0x^2 + 1x^3 \\
+ \quad 2 + 3x + 7x^2 + 4x^3 \\
\hline
3 + 1x + 7x^2 + 5x^3
\end{array}
$$

isomorphic to

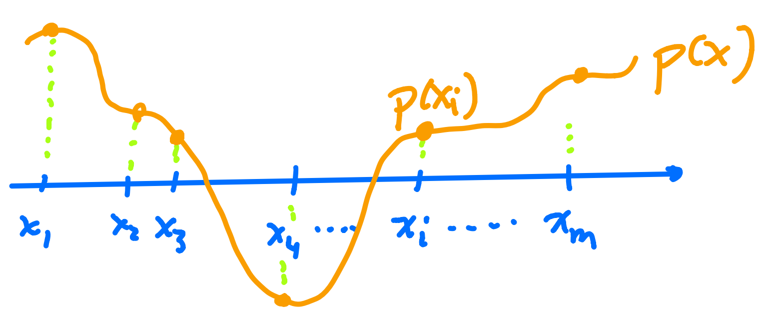
$$W = \begin{pmatrix} 3 \\ 1 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 7 \\ 4 \end{pmatrix}$$

//

The Vandermonde Matrix is an $m \times n$ matrix of constant entries.

$$A = \begin{bmatrix} x_1^0 & x_1^1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ \vdots & & & & & \vdots \\ x_m^0 & x_m^1 & x_m^2 & x_m^3 & \cdots & x_m^{n-1} \end{bmatrix}.$$

It arises in polynomial interpolation & polynomial approximation.



$p(x) \in \mathcal{P}^{n-1}(x)$

$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_{n-1} x^{n-1}$

$p(x_i) = p(x)\Big|_{x = x_i} = (Ac)_i$

$$A = \begin{bmatrix} x^0 & x^1 & x^2 & \dots & x^{n-1} \end{bmatrix} \quad c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$p(x_i) = c_0 x_i^0 + c_1 x_i^1 + c_2 x_i^2 + \dots c_{n-1} x_i^{n-1}$ //

Matrix times Matrix $B = AC$

$B \in \mathbb{C}^{\ell \times n}$

$A \in \mathbb{C}^{\ell \times m}$

$C \in \mathbb{C}^{m \times n}$

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj} \quad \begin{cases} 1 \le i \le \ell \\ 1 \le j \le n \\ 1 \le k \le m \end{cases}$$

in index notation.

We will think of matrix matrix multiplication as representing each column of B as a linear combination of columns of A:

Take b_2, the 2nd column of B:

$$\ell\left[\; b_2 \;\right] = \ell\,\overbrace{\left[\quad A \quad\right]}^{m}\left[\; c_2 \;\right] m$$

Take the j^{th} column of B:

$$\left[\; b_j \;\right] = \left[\quad A \quad\right]\left[\; c_j \;\right]$$

for $j = 1, 2, \ldots, n$.

So we think of $B = AC$ as representing n matrix-vector products

$$\ell\,\overbrace{\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}}^{n} = \ell\,\underbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{bmatrix}}_{m}\;\overbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ & & & \\ & & & \end{bmatrix}}^{n} m$$

In index form:

$$b_j = Ac_j = \sum_{k=1}^{m} c_{kj}\, a_k$$