MATH 342 MIDTERM EXAM

NAME:

Instructions: You must include all the steps in your derivations/answers. Reduce answers as much as possible, but use *exact arithmetic*. Write neatly.

1. (20 %)

   (a) Let X and Y be subspaces of V, a vector space. Prove that $X \cap Y$ is a subspace of V.

   (b) Let X be a subspace of V. Let $\mathbf{v} \in V$, and $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$ that $\mathbf{x} + \mathbf{v} \notin X$.

   Answer: See Homework 2

2. (10 %) Find a 2×3 system (2 equations, 3 unknowns) such that its general solution is of the form

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Answer: See Homework 3

3. (10 %) For what values of h and k is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

Answer: Adding 3 times the first row to the second one, we find the second equation is $3h + k = 0$. For consitency we thus require $h = -k/3$, where $k \in \mathbb{R}$.

4. (20 %) Determine which of the following polynomials belong to the subspace of $\mathcal{P}$ spanned by $x^3 + 2x^2 + 1$, $x^2 - 2$, and $x^3 + x$. Give an argument to support your answer.

   (a) $x^2 - x + 3$.

   (b) $x^2 - 2x + 1$.

   (c) $x - 5$.

   Answer: (a) and (c) are, but (b) is not. To see this we use an isomorphism to $\mathbb{R}^n$, as follows: let

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

   and $\mathbf{X} = (x^3, x^2, x, 1)^\top$. The span of A is

$$\text{span}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} := (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})\,.$$

   Case (a)

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \mathbf{a_1} - 1\mathbf{a_2} - 1\mathbf{a_3}.$$

   Case (b). Can't form a linear combination that yields $\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}^\top$.

   Case (c).

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -5 \end{pmatrix} = -1\mathbf{a_1} + 2\mathbf{a_2} + 1\mathbf{a_3}$$

5. (20 %) Let U be a subspace of $\mathbb{R}^5$, defined by

$$U = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, \quad \text{and} \quad x_3 = 7x_4\right\}.$$

(a) Find a basis for U.

(b) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Answer:

Form

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5) : x_2, x_4, x_5 \in \mathbb{R}\},$$

from which one can suggest the following basis for U: $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$.
to find a W such that $\mathbb{R}^5 = U \oplus W$, we could choose W to have a basis $(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$.

6. (10 %) Suppose

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -\frac{3}{2} \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Suppose

$$B = A - 2I_2,$$

where I_2 is the 2×2 identity matrix. What are the eigenvalues of B and how do these relate to those of A?

Answer: The eigenvalues of A are 1 and $-3/2$. The vectors are $\mathbf{v}_1 = [1 \quad 0]^T$ and $\mathbf{v}_2 = [2/5 \quad 1]^T$. Since B is obtained by adding 2 to the diagonals of A the direction of the vectors does not change and the eigenvalues simply change by 2: the eigenvalues of B are -1 and $-5/2$.

7. (10 %) Find the column space, row space, the null space of A and the null space of A^T, for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Answer: see class notes.