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HW6

(1) Show that A^+A has the same nullspace as A where $A \in \mathbb{F}^{m \times n}$

if $Ax = 0$ then $A^+Ax = 0$ ∴ vectors in the nullspace of A are in the nullspace of A^+A.

To go in the other direction, start by supposing $A^+Ax = 0$, take the inner product with x and show that $Ax = 0$:

$x^T A^+ A x = 0$ or $\|Ax\|^2 = 0$ or $Ax = 0$

the 2 nullspaces are identical. In particular if A has LI columns (and $x = 0$ is only element in nullspace) $\Rightarrow$ same is true for A^+A:

If A has LI cols $\Rightarrow$ A^+A is square, symmetric, invertible.

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(2) let $A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$ $b \in \mathbb{R}^m$

s.t. $Ax = Pb + r = b$

where $\underset{\sim}{r} \perp \mathrm{col}(A)$

$P = A(A^TA)^{-1}A^T$ is the projection matrix

(a) Show that $P^2 = P$

(b) Show that $P^T = P$

(c) Conversely, any symmetric matrix with $P^2 = P$ represents a projection

(a) since $Pr = 0$ and $P(Pb) = P^2 b = Pb + Pr = Pb$

also: $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$

(b) $P^T = (A^T)^T\left((A^TA)^{-1}\right)^TA^T = A\left((A^TA)^T\right)^{-1}A^T = A(A^TA)^{-1}A^T = P$

(c) since $\underset{\sim}{r} \perp Pb \Rightarrow (1-P)b = \underset{\sim}{r}$

for any vector Pc, the inner product must be zero:

$(b-Pb)^TPc = b^T(1-P)^TPc = b^2(P-P^2)c = 0$

since $P = P^2$.

$\therefore (b-Pb) = \underset{\sim}{r}$ is orthogonal to Pb

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(3) Find the projection of $\underset{\sim}{b}$ onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Split $\underset{\sim}{b} = p + q$, where $p \in col(A)$ and $q \perp col(A)$.

To which of the 4 subspaces of A does q belong to?

$$p = Pb = A\hat{x} = A(A^TA)^{-1}A^Tb = \begin{pmatrix} 2.0909 \\ -1.2727 \\ 5.9091 \end{pmatrix}$$

$$q = (I-P)b = b - Pb = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 2.0909 \\ -1.2727 \\ 5.9091 \end{bmatrix} = \begin{bmatrix} -1.0909 \\ 3.2727 \\ 1.0909 \end{bmatrix}$$

$q \in Null(A^T)$ i.e. $A^Tq = A^T(b - A\hat{x}) = 0$

$A^Tq = 0$

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$S \in \mathcal{L}(W,U),\ T \in \mathcal{L}(V,W)$

(4) Suppose S and T ~~$\in \mathcal{L}(V)$~~ are self adjoint.

Prove that ST is self adjoint iff $ST = TS$

Generally: $\langle v, (ST)^{+}u \rangle = \langle STv, u \rangle = \langle Tv, S^{+}u \rangle$

$$= \langle v, T^{+}(S^{+}u) \rangle$$

suppose $ST = TS$:

$$\langle v, (ST)^{+}u \rangle = \langle STv, u \rangle = \langle TSv, u \rangle$$

$$= \langle Sv, T^{+}u \rangle = \langle v, S^{+}(T^{+}u) \rangle$$

$$= \langle v, (ST)^{+}u \rangle$$

Suppose $K = ST$, if $K = K^{+}$

$$\langle Kv, w \rangle = \langle v, Kw \rangle$$

$$\langle STv, w \rangle = \langle v, (ST)w \rangle$$

$$= \langle Sv, Tw \rangle = \langle TSv, w \rangle$$

$\therefore$ if $ST = TS$ then $K = K^{+}$

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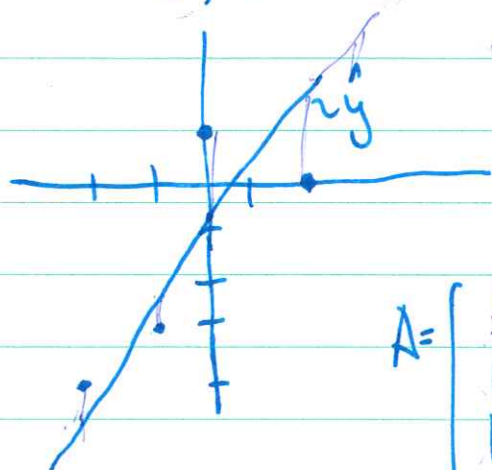
(5) Find best straight line fit to $(-2,-4)$, $(-1,-3)$, $(0,1)$, $(2,0)$



$y = mx + b$ is the model

let $\cancel{z = \begin{pmatrix} m \\ b \end{pmatrix}}$ $z = \begin{pmatrix} b \\ m \end{pmatrix}$

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \therefore \quad Az = \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1.2286 \\ 1.0857 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \qquad \hat{z} = (A^TA)^{-1}A^Tb = \cancel{\begin{bmatrix} -0.8644 \\ 0.8475 \end{bmatrix}}$$

$\cancel{\hat{y} = 0.8475x - 0.8644}$ $\hat{y} = -1.22086 + 1.0857x$

$$\cancel{\|r\| = \left\{ \sum_{i=1}^{4} (y_i - \hat{y}(x_i))^2 \right\}^{1/2} = \left\{ (-2.559+4)^2 + (-1.7119+3)^2 + (-0.8644+1)^2 + (0-0.8305)^2 \right\}^{1/2}} = 2.81$$

$$\|r\| = \left\{ \sum_{i=1}^{4} (y_i - \hat{y}(x_i))^2 \right\}^{1/2} = 2.5857$$

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(6) Fit a plane $z = a + bx + cy$ to $(1,1,3)$, $(0,3,6)$, $(2,1,5)$, $(0,0,0)$

(a) Find 4 equations with 3 unknowns (a, b, c) such that the plane passes through all points (system does not have a solution)

(b) Find least squares solution

let $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$AX = B = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}$

$\hat{X} = (A^T A)^{-1} A^T B = \begin{bmatrix} -0.12 \\ 1.46 \\ 2.02 \end{bmatrix}$

$\hat{z} = A\hat{X} \Rightarrow \hat{z} = -0.12 + 1.46x + 2.02y$

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(7) $f(x) = (Ax-b)^T(Ax-b)$

$$\frac{df}{dx} = A^T(Ax-b) = 0$$

or $A^TA\hat{x} = A^Tb$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$7' : A'$