HW5

① Explain why each of the following is not an inner product on a given vectorspace:

(a) $(\underset{\sim}{x}, \underset{\sim}{y}) = x_1 y_1 - x_2 y_2$ on $\mathbb{R}^2$

among other things $(\underset{\sim}{x}, \underset{\sim}{x}) \not\geq 0 \;\forall\, \underset{\sim}{x} \in \mathbb{R}^2$

(b) $(A, B) = tr(A+B)$ on $\mathbb{R}^{2\times 2}$

$(A, A) = tr(A+A) = 2\,tr(A)$ which could be < 0 if $tr(A) < 0$.

(c) $\langle f, g \rangle = \int_0^1 \frac{df}{dt} \bar{g}\, dt$ on $\mathcal{P}(\mathbb{F})$: first take $\mathcal{P}(\mathbb{R})$:

suppose $f' < 0$, then $\langle f, f \rangle = \int_0^1 f' \bar{f}\, dt = \int_0^1 \frac{d}{dt} f^2\, dt = f^2 \Big|_0^1 \geq 0$

so no problem with positivity

$\langle f, f \rangle = 0$ iff $f = 0$, no problem with definiteness

conjugate symmetry is ok too: $\langle f, g \rangle = \langle f, g \rangle$

If $\mathcal{P}(\mathbb{F}) = \mathcal{P}(\mathbb{C})$ conjugate symmetry would fail, so will additivity and homogeneity.

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(2) Find all vectors $\perp$ to $(1,1,1,1)^T$ and $(1,2,3,4)^T$ in $\mathbb{R}^4$:

first, build 1 vector $\perp$ to 2 first 2:

$$\left.\begin{array}{l}(1,1,1,1)\cdot(a,b,c,d)=0\\(1,2,3,4)\cdot(a,b,c,d)=0\end{array}\right\}\begin{array}{l}b=-2c-3d\\a=c+2d\end{array}$$

set $c=0,\ d=1 \Rightarrow (2,-3,0,1)$

next, build another $\perp$ to first 3:

$$(1,1,1,1)\cdot(\alpha,\beta,\gamma,\theta)=0$$
$$(1,2,3,4)\cdot(\alpha,\beta,\gamma,\theta)=0$$
$$(2,-3,0,1)\cdot(\alpha,\beta,\gamma,\theta)=0$$

$$\alpha=-\frac{1}{7}\gamma,\ \beta=-\frac{2}{7}\gamma,\ \theta=-\frac{4}{7}\gamma$$

$$\therefore \left(-\frac{1}{7},-\frac{2}{7},1,-\frac{4}{7}\right)$$

③ Find an orthonormal basis for the following subspace of $\mathbb{R}^4$:

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; x-y-z+w=0 \;\&\; x+z=0 \right\}$$

$x=-z$

$y=-2z+w$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = z\underset{\sim}{Q}_1 + w\underset{\sim}{Q}_2$$

Note that $\underset{\sim}{Q}_1 \cdot \underset{\sim}{Q}_2 = 0$

$$q_1 = \frac{\underset{\sim}{Q}_1}{\|\underset{\sim}{Q}_1\|} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{1^2+2^2+1^2+0}} = \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \\ 0 \end{pmatrix}$$

$$q_2 = \frac{\underset{\sim}{Q}_2}{\|\underset{\sim}{Q}_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(4) Show that square of average $\leq$ to average of squares: $\{a_i\}_{i=1}^{n}$

$$\bar{a} = \frac{a_1 + a_2 + \cdots a_n}{n} \quad \text{so} \quad \bar{a}^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} a_i\right)^2$$

and $\sum_{i=1}^{\infty} \frac{a_i^2}{n}$ $\therefore$ show that $\frac{1}{n^2}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} a_i^2$

$*$ $\|u\|\,\|v\| \geq |\langle u, v\rangle|$ is Cauchy Shwartz

now, take $\underset{\sim}{u} = (a_1, a_2 \ldots, a_n)$ take $\underset{\sim}{v} = (1, 1, 1 \ldots, 1)$

Square of $(*)$ yields

$$|u_1 v_1 + u_2 v_2 \cdots u_n v_n|^2 \leq \sum_{i=1}^{n} u_i^2 \sum_{i=1}^{n} v_i^2 \text{, now, use}$$

$$\left|\sum_{i=1}^{n} a_i\right|^2 \leq n \sum_{i=1}^{n} |a_i|^2$$

Hence $\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2$

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(5) Suppose $T \in \mathbb{R}^{n \times n} \in \mathcal{L}(V)$ s.t. $\|T\underset{\sim}{v}\| \leq \|v\|$ for every $\underset{\sim}{v} \in V$, Prove that

$T - \sqrt{2}\, I$ is invertible

$$\|T\underset{\sim}{v}\| = \|\lambda \underset{\sim}{v}\| \leq \|v\| \quad \therefore -1 \leq \lambda \leq 1$$

Hence $\cancel{\lambda \neq \sqrt{2}}$ $\det[T - \lambda I] = 0$

only if $|\lambda| \leq 1$

but $\sqrt{2} > 1$

so $\det|T - \sqrt{2} I| \neq 0$

Matrix is invertible iff $\det(\text{Matrix}) \neq 0$

$\therefore T - \sqrt{2} I$ is invertible.

(6) Apply G-S to $\begin{pmatrix}1\\2\\-2\end{pmatrix}, \begin{pmatrix}1\\-1\\4\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}$

$$\underline{q}_1 = \frac{\underline{Q}_1}{\|\underline{Q}_1\|} = \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$$\underline{Q}_2 = \begin{pmatrix}1\\-1\\4\end{pmatrix} - \underline{q}_1 \cdot \begin{pmatrix}1\\-1\\4\end{pmatrix} \underline{q}_1 = \begin{pmatrix}1\\-1\\4\end{pmatrix} - \frac{9}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}0\\-3\\6\end{pmatrix}$$

$$\underline{q}_2 = \frac{1}{\sqrt{45}}\begin{pmatrix}0\\-3\\6\end{pmatrix}$$

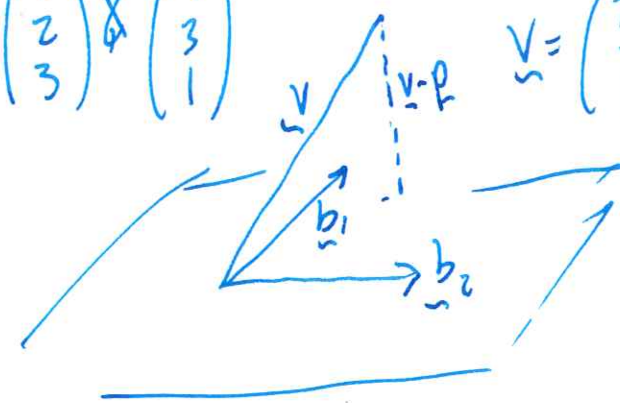
$$Q_3 = \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{1}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix} - \frac{1}{45}\begin{pmatrix}0\\-3\\6\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}0\\-3\\6\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{2}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix} - \frac{\cancel{3}}{\cancel{45}_{15}}\begin{pmatrix}0\\-3\\6\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}2/9\\4/9\\-4/9\end{pmatrix} - \begin{pmatrix}0\\-1/5\\2/5\end{pmatrix} = \begin{pmatrix}16/9\\5/9\\13/9\end{pmatrix} + \begin{pmatrix}0\\1/5\\-2/5\end{pmatrix}$$

$$Q_3 = \begin{pmatrix}\frac{16}{9}\\ \frac{26}{45}\\ \frac{63}{45}\end{pmatrix} \qquad q_3 = \frac{1}{\left[\left(\frac{16}{9}\right)^2 + \left(\frac{26}{45}\right)^2 + \left(\frac{63}{45}\right)^2\right]^{\frac{1}{2}}}\begin{pmatrix}16/9\\26/45\\63/45\end{pmatrix}$$

(7) Find the distance from $\begin{pmatrix}2\\3\\1\end{pmatrix}$ to the subspace spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}$ & $\begin{pmatrix}1\\3\\1\end{pmatrix}$



$\underset{\sim}{v} = \begin{pmatrix}2\\3\\1\end{pmatrix}$

$(\underset{\sim}{v}-\underset{\sim}{p})\cdot\underset{\sim}{b}_1 = 0$

$(\underset{\sim}{v}-\underset{\sim}{p})\cdot\underset{\sim}{b}_2 = 0$

$\begin{cases}\underline{v}\cdot\underline{b}_1 = \underline{p}\cdot\underline{b}_1\\ \underline{v}\cdot\underline{b}_2 = \underline{p}\cdot\underline{b}_2\end{cases}$ but $\underset{\sim}{p} = \alpha\underset{\sim}{b}_1 + \beta\underset{\sim}{b}_2$ since it lives in the span$(\underline{b}_1, \underline{b}_2)$

$\underset{\sim}{v}\cdot\underset{\sim}{b}_1 = \alpha\,\underline{b}_1\cdot\underline{b}_1 + \beta\,\underline{b}_1\cdot\underline{b}_2$

$\underset{\sim}{v}\cdot\underset{\sim}{b}_2 = \alpha\,\underline{b}_1\cdot\underline{b}_2 + \beta\,\underline{b}_2\cdot\underline{b}_2$

$11 = \alpha 14 + \beta 10$; $12 = 10\alpha + 11\beta$

$\alpha = 0.0185 \qquad \beta = 1.0741$

$\underset{\sim}{v}-\underset{\sim}{p} = \underset{\sim}{v} - \alpha b_1 - \beta b_2 = \begin{bmatrix}0.9074\\-0.2593\\-0.1296\end{bmatrix}$

the distance is $\sqrt{(\underline{v}-\underline{p})\cdot(\underline{v}-\underline{p})} = 0.9526$

(8) True or false: if E is subspace of V then $\dim E + \dim E^{\perp} = \dim V$

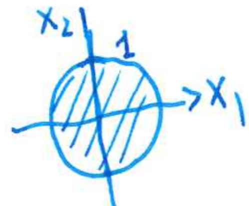
True. $V = E \oplus E^{\perp}$, i.e. $E \cap E^{\perp} = \emptyset$, so $\dim V = \dim E + \dim E^{\perp}$

(9) let $\|\underset{\sim}{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad 1 \le p < \infty$

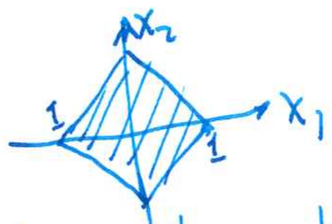
$\|x\|_\infty = \max\{|x_k| \; k=1,2..n\}$

In $\mathbb{R}^2$ $\quad B_p = \{\underset{\sim}{x} \in \mathbb{R}^2 \mid \|x\|_p \le 1\}$

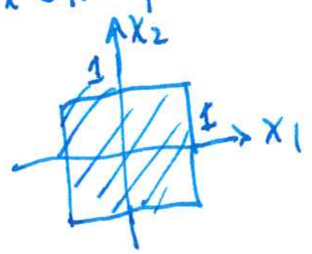
So $B_2 = \{x \in \mathbb{R}^2 \mid \|x\|_2 \le 1\} = \{\underset{\sim}{x} \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} \le 1\}$

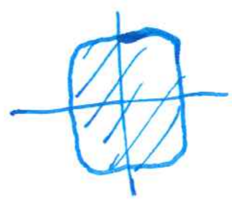


$B_1 = \{x \in \mathbb{R}^2 \; | |x_1|^2 + |x_2|^2 \le 1\}$



$B_\infty = \{x \in \mathbb{R}^2 \mid \max|x_1|,|x_2| \le 1\}$



B_p for $p > 2$  superellipses