HW5

1. Explain why each of the following is **not** an inner product on a given vector space

   (a) $(\underset{\sim}{x}, \underset{\sim}{y}) = x_1y_1 - x_2y_2$ on $\mathbb{R}^2$

   (b) $(A, B) = \text{trace}(A+B)$ on the space of $\mathbb{R}^{2\times 2}$ matrices

   (c) $\langle f, g\rangle = \int_0^1 f'(t)\overline{g(t)}\,dt$ on the space of polynomials. Here, $f' = \frac{df}{dt}$

2. Find all vectors in $\mathbb{R}^4$ orthogonal to $(1,1,1,1)^T$ and $(1,2,3,4)^T$

3. Find an orthonormal basis for the following subspace of $\mathbb{R}^4$:

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle|\ x - y - z + w = 0 \text{ and } x + z = 0 \right\}$$

4. Show that the square of an average is less than or equal to the average of the squares. More precisely show that if $a_1, a_2, \dots, a_n \in \mathbb{R}$ then the square of the average of all the a's is less than or equal to the average of $a_1^2, \dots, a_n^2$.

5. Suppose $T \in \mathbb{R}^{n\times n} \in \mathcal{L}(V)$ such that $\|Tv\| \le \|v\|$, for every $\underset{\sim}{v} \in V$. Prove that $T - \sqrt{2}I_n$ is invertible.

(6) Apply Gram-Schmidt to $(1,2,-2)^T$, $(1,-1,4)^T$, $(2,1,1)^T$

(7) Find the distance from the vector $(2,3,1)^T$ to the subspace spanned by $(1,2,3)^T$ and $(1,3,1)^T$. Note: find distance, no need to find orthogonal projection.

(8) True or false: if E is a subspace of V then $\dim E + \dim E^{\perp} = \dim V$? Justify answer.

(9) Let $\|\cdot\|_p$, where $1 \leq p < \infty$ be the l_p norm. It is defined, in $\mathbb{R}^n$, as $\|\underset{\sim}{x}\|_p = \left(|x_1|^p + |x_2|^p + \dots |x_n|^p\right)^{1/p} = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$

Note $\|\underset{\sim}{x}\|_2$ corresponds to the norm obtained from the inner product.

There is also $\|\underset{\sim}{x}\|_\infty = \max\{|x_k| : k = 1, 2 \dots, n\}$.

Consider $\mathbb{R}^2$ space. For $\underset{\sim}{x} \in \mathbb{R}^2$, the "unit ball" B_p in the norm $\|\cdot\|_p$ is defined as

$$B_p = \{\underset{\sim}{x} \in \mathbb{R}^2 : \|\underset{\sim}{x}\|_p \leq 1\}$$

Find the shapes of B_1, B_2, B_∞