# HW4

(1) Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w,z) = (z,w)$
find e'values & e'vectors of $T$

$T(w,z) = \lambda(w,z)$ or $z = \lambda w$, $w = \lambda z$

hence $w = \lambda^2 w \therefore \lambda^2 = 1 \Rightarrow \lambda_{1,2} = \pm 1$. Associated with

$\lambda = +1$ $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = -1$ $\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(2) Prove that $I - T$ is invertible under certain conditions.

$T \in \mathcal{L}(V)$ $\exists$ $n > 0$ integer s.t. $T^{n+1} = 0$

and that

$(I-T)^{-1} = I + T + \cdots T^{n}$

$$(I-T) = I - SDS^{-1} = S(I-D)S^{-1}$$

$$(I-T)^{-1} = S^{-1}(I-D)^{-1}S$$

$$= S^{-1} \begin{bmatrix} \frac{1}{1-\lambda_1} & & & \\ & \frac{1}{1-\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{1-\lambda_n} \end{bmatrix} S$$

since $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$, and assuming $|x| \neq 1$
and $x^{n+1} = 0$

$$(I-T)^{-1} = S^{-1} \begin{bmatrix} \sum_{k=0}^{n} \lambda_1^k & & & \\ & \sum \lambda_2^k & & \\ & & \ddots & \\ & & & \sum_{k=0}^{n} \lambda_n^k \end{bmatrix} S$$

$k=0 \qquad S^{-1} I S = I$

$k=1 \qquad S^{-1} D S = T$

$k=2 \qquad S^{-1} D^2 S = T^2$

$\vdots$

$k=n \qquad S^{-1} D^n S = T^n$

$\therefore (I-T)^{-1} = I + T + \cdots T^n$ require that $|\lambda_i| \neq 1$ for $i=1,\dots,n$.

(3) Find evalues & characteristic polynomial to

$$\begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & \pi & 43 & 2 \\ 0 & 0 & 16 & 1 \\ 0 & 0 & 0 & 54 \end{pmatrix}$$

since triangular

$p(\lambda) = (\lambda-2)(\lambda-\pi)(\lambda-16)(\lambda-54) = 0$

with e'v : $2, \pi, 16, 54$

(4) An operator $A$ is nilpotent if $A^k = \emptyset$ for some $k$.
Prove that if $A$ is nilpotent then $\sigma(A) = \{0\}$ is the only eigenvalue of $A$:

$A\underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq 0$

by induction

$A^k \underline{x} = \lambda^k \underline{x} = 0$

$A^{k-1} A\underline{x} = \lambda^{k-1} \lambda \underline{x} = 0$

$\vdots$

so $\lambda = 0$

⑤ let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ find $A^{1000}$

Diagonalize $\quad A = SDS^{-1}$

$$\det(A - \lambda I) = (4-\lambda)(2-\lambda) - 3 = 0$$
$$= 8 - 2\lambda - 4\lambda + \lambda^2 - 3 = \lambda^2 - 6\lambda + 5 = 0$$
$$= (\lambda - 5)(\lambda - 1) \quad \Rightarrow \lambda_{1,2} = 5, 1$$

with e'functions $\underline{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$S = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

So $A^{1000} = S \begin{pmatrix} 1 & 0 \\ 0 & 5^{1000} \end{pmatrix} S^{-1}$

⑥ Diagonalize $A = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ So $\lambda = 2, 2, 4$

$$S = \begin{pmatrix} 1 & 0 & 0.8018 \\ 0 & 1 & 0.5345 \\ 0 & 0 & 0.2673 \end{pmatrix} \qquad D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 3.7417 \end{pmatrix}$$

(7) Convergent matrix : let $A \in F^{n \times n} = (a)_{ij}$

$A$ is "convergent" if $\lim_{k \to \infty} a_{ij} = 0$ for each $i, j = 1, 2, \ldots n$.

(a) Take $A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$, $A^2 = \begin{pmatrix} \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{pmatrix}$, $A^n = \begin{pmatrix} \frac{1}{4^n} & \frac{1}{2^n} \\ 0 & \frac{1}{4^n} \end{pmatrix}$

in ~~general $A^k =$~~ $\lim_{k \to \infty} a_{ij}^k = 0 \quad i, j = 1, 2 \ldots n$.

(b) Since $\rho(A) \equiv \max \{ |\lambda_i| \}_{i=1}^{n}$

Show that $\rho(A) = \| A^k \|^{1/k}$, use $A \underset{\sim}{v} = \lambda \underset{\sim}{v} : \underset{\sim}{v} \neq 0$

$$|\lambda|^k \|v\| = \| \lambda^k \underset{\sim}{v} \| \leq \| A^k \underset{\sim}{v} \| \leq \| A^k \| \, \|v\|$$

$\therefore \rho(A) \leq \| A^k \|^{1/k}$

(c) if $\rho(A) < 1$ then $\lim_{k \to \infty} A^k = 0$

$$0 = \lim_{k \to \infty} A^k \underset{\sim}{v} = \lim_{k \to \infty} (A^k \underset{\sim}{v}) = \lim_{k \to \infty} \lambda^k \underset{\sim}{v} = \underset{\sim}{v} \lim_{k \to \infty} \lambda^k$$

take $\lambda$ to be $\rho(k)$ so $\lim_{k \to \infty} [\rho(k)]^k = 0$ //

(8) $AP_0 = \begin{pmatrix} 0.25 \\ 0.1 \\ 0.4 \\ 0.25 \end{pmatrix}$

$AP_1 = A^2P_0 = \begin{pmatrix} 0.42 \\ 0.02 \\ 0.17 \\ 0.39 \end{pmatrix}$

$$\lim_{n\to\infty} A^n P_0 = \lim_{n\to\infty} (Q D Q^{-1})^n P_0$$

$$= \begin{pmatrix} 1 & \frac{27}{56} & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{29}{56} & \frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{59}{112} \\ 0 \\ 0 \\ \frac{53}{112} \end{pmatrix}$$

here $D = \begin{pmatrix} 1 & & & \\ & 0.2 & & \\ & & 0.3 & \\ & & & 1 \end{pmatrix}$ $\lim_{n\to\infty} D^n = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$