HW4

① Let $T \in \mathcal{L}(\mathbb{R}^2)$ s.t. $T(w,z) = (z,w)$
Find eigenvalues/vectors of T.

② Prove that, if $T \in \mathcal{L}(V)$ and $\exists$ $n>0$, integer, s.t. $T^{n+1} = 0$

$$(I-T)^{-1} = I + T + T^2 + \cdots T^n$$

hint: Use the geometric series.

(b) What condition must be satisfied with regard to the magnitude of the eigenvalues of T?

③ Find the characteristic polynomial and eigenvalues of
$$\begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & \pi & 43 & 2 \\ 0 & 0 & 16 & 1 \\ 0 & 0 & 0 & 54 \end{pmatrix}$$

④ An operator A is "nilpotent" if $A^k = 0$ for some $k>0$
Prove that if A is nilpotent then $\sigma(A) = \{0\}$
i.e. the only eigenvalue in the spectrum of A is 0.

(5) let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ find A^{1000}

(6) Diagonalize $A = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

(7) Let $A \in \mathbb{F}^{n \times n}$, with entries a_{ij} : $i, j = 1, 2, .. n$.

A is said to be "convergent" if $\lim_{k \to \infty} a_{ij} = 0$ for each $i, j = 1, 2 \ldots n$.

(a) Take $A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$, find A^4, find A^n

prove that A is convergent.

(b) Define $\rho(A) \equiv$ largest eigenvalue of A in modulus:

$$\rho(A) \equiv \max_{1 \le i \le n} \{|\lambda_i|\}$$, its a non-negative scalar.

Show that $\rho(A) = \|A^k\|^{1/k}$. Use $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ with $\underset{\sim}{v} \neq 0$.

The symbol $\|A^k\|$ is the "induced matrix norm" of A^k. Take it as some estimate of the size of A^k.

(c) if $\rho(B) < 1$ Show that

$$\lim_{k \to \infty} B^k = 0$$

(8) Some events are described probabilistically... by a number from 0 to 1, representing the normalized frequency at which it is observed to occur. (The number 0.5, for example, means that a certain outcome occurs ~~[illegible]~~ 50% of the time).

Let's assume that there are a finite number of outcomes for a certain event. We construct a "probability vector" P which describes the probability of being in each state or outcome. The column sum will be 1.

A probability vector can be made to evolve via a linear transformation, forming a "Markov Process" which is specifically called a "Markov Chain".

Let $P_n \in U^M$, where $U = [0,1]$
, n is the sequence index.
, M is the dimension of the vector P_n.

The Markov Chain is specified by

$$\begin{cases} P_{n+1} = A P_n & n = 0, 1, 2 \ldots \\ P_0 \text{ given} \end{cases}$$

$A \in U^{M \times M}$, the column sum of A is 1 is called the "transition matrix"

P_n is called the "state" at "time" n. The chain is called "Markovian" because the state of P at $n+1$ is entirely described by A and P_n. (A non-Markovian, could be one where $P_n, P_{n-1} \dots$ and many other states determines P_{n+1}).

A common question in Markov Chains is to determine the "long-time limit

$$\lim_{n\to\infty} P_n = \quad = \lim_{n\to\infty} A P_{n-1} = \lim_{n\to\infty} A^2 P_{n-2} \cdots =$$

$$= \lim_{n\to\infty} A^n P_0$$

(from "Linear Algebra" by Friedberg, Insel, Spence)

A community college wants to determine the likelihood of students in various classes graduate. Of

(a) sophomores, 40% will graduate ( )
30% remain sophomores
30% quit

(b) freshman, 10% will graduate ( )
50% become sophomores
20% remain freshman
20% quit.

There are presently 50% freshmen and 50% sophmores.

So $P_0 = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$ ← graduated, ← freshmen, ← sophomore, ← quit

Data from (a) & (b) will be assumed to be the rates and don't change from year n, to year n+1.

The school would like to know

(1) the % of present students who will graduate
% of " " who " freshmen
% " " " " " sophmores
% " " " " " quit.
(BY NEXT YEAR)

(2) Same question, but (BY TWO YEARS TIME)

(3) Same question, but (AFTER MANY YEARS)

We build transition matrix

$$A = \begin{pmatrix} 1 & 0.1 & 0.4 & 0 \\ 0 & 0.5 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.3 & 1 \end{pmatrix}$$

* Explain the transition matrix

* Answer question (1), i.e. find $P_1 = AP_0$ & interpret

* Answer question (2), i.e. find $P_2 = AP_1 = A^2 P_0$

* Answer (3), i.e. find $\lim_{n \to \infty} P_n$ and interpret

For this last step, find a decomposition

$$A = QDQ^{-1}$$

where D is a diagonal matrix, Q an invertible matrix.