HW2

(1) Show that a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be done by matrix multiplication. Verify that T is a linear transformation.

let $V = \mathbb{R}^n$ $W = \mathbb{R}^m$ be the 2 vector spaces.

(If $n = m$ then $V = W$)

$T: V \to W$

vectors in W are of the form $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$

we write $y = \hat{e}_1 x_1 + \hat{e}_2 x_2 \cdots \hat{e}_m x_m = \sum_{k=1}^{n} x_k \hat{e}_k$

$\hat{e}_k$ are

$\hat{a}_1 = T(\hat{e}_1)$, $\hat{a}_2 = T(\hat{e}_2) \cdots$ $\hat{a}_n = T(\hat{e}_n)$

$y = T(\underset{\sim}{x}) = \sum_{k=}^{n} T(x_k \hat{e}_k) = \sum_{k=}^{n} x_k \hat{a}_k$

where $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

So $y \in W$ and $\underset{\sim}{x} \in V$. Represent all of the vectors $\hat{a}_i$ as columns of $A = (\hat{a}_1\ \hat{a}_2 \cdots a_n) \equiv \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & & & a_{2,n} \\ a_{3,1} & & & \vdots \\ \vdots & & & \vdots \\ a_{m,1} & & & a_{m,n} \end{pmatrix}$

then $y = A\underset{\sim}{x} = \sum_{j=1}^{n} a_{ij} x_j$

(2) Find a basis for the vector space $\mathbb{R}^{3\times2}$. What is the dimension of $\mathbb{R}^{3\times2}$ (matrices of reals, size 3x2)

An element of $\mathbb{R}^{3\times2}$ is

$$A \equiv \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{pmatrix} \quad \text{where } a_i \in \mathbb{R}$$

intuitively we have 6 degrees of freedom, so we need

$\dim(\mathbb{R}^{3\times2}) = 6$

so
$$A = a_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + a_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$+ a_5\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + a_6\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \equiv a_i A_i$$

or A can be written as ~~any~~ a linear combination of the A_i's

(3) Prove that the set of $n \times n$ real symmetric matrices is a subspace of $\mathbb{R}^{n\times n}$ ~~Prove~~

A is symmetric if $A^T = A$

and anti-symetric if $A^T = -A$

(a) Show that $A \in \mathbb{R}^{n\times n}$ can be written as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$A = (a_{ij}) \quad i=1,\dots,n \;, j=1,\dots,n$

$a_{ji} = A^T$ and $-a_{ij} = -A$

$\frac{1}{2}(A+A^T) = \frac{1}{2}(a_{ij}+a_{ji})$

$\frac{1}{2}(A-A^T) = \frac{1}{2}(a_{ij}-a_{ji})$

so adding these 2

$\frac{1}{2}a_{ij} + \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} - \frac{1}{2}a_{ji} = a_{ij} = A$

To show that $A^T = A$ matrices are a subspace of A, the $\mathbb{R}^{n\times n}$

closed under addition: $(A+B)^T = A^T + B^T$

closed under scaler multiplication $(cA)^T = cA^T$

assuming $A^T = A$ then $(A+B)^T = A+B$

" $A^T = A$ then $(cA)^T = cA$

assuming $A^T = -A$ then $(A+B)^T = -A-B$

" " then $(cA)^T = -cA$

Both subspaces have $\emptyset$ member: $\emptyset \in \mathbb{R}^{n\times n}$

$\emptyset^T = \emptyset$ & $\emptyset^T = -\emptyset$, respectively

To check that $\frac{1}{2}(A+A^T)$, ~~is~~ symmetric & $\frac{1}{2}(A-A^T)$ ~~is~~ antisymmetric, by direct sum we can verify that the only matrix is both symmetric & anti-symmetric is $\emptyset$

$A = A^T = -A$ so $2A = \emptyset$ so $A = \emptyset$.

(14) The set $\mathbb{C}$ of complex numbers can be cannonically identified with the space $\mathbb{R}^2$ by treating $\mathbb{C} \ni x + iy = z$ as a column $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

(a) Show that if we treat this as a complex vector space, multiplication by $\alpha \in \mathbb{C}$ is a linear transformation in $\mathbb{C}$. What is the matrix?

$$w = T(z) \equiv \alpha z \qquad \alpha \in \mathbb{C}$$

$$w = (a+ib)(x+iy) = ax + i(bx+ay) - by$$

so $T = (1)$ the 1×1 matrix with entry 1

(b) Treat $\mathbb{C}$ as $\mathbb{R}^2$, find the linear transformation that represents multiplication of $x+iy$ by $\alpha = a+ib$

$$\alpha z \text{ is } \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}$$

(3) For each linear transformation, find the matrix transformation

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$, defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-5y \\ 7y \end{pmatrix}$

by inspection $A = \begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix} \in \mathbb{R}^{2\times 3}$

(b) $T: P_n \to P_{n-1}$, $Tf(t) = \frac{df}{dt}$, find the matrix with respect to the basis $(1, t^1, t^2, t^3, \ldots, t^n)$

so if $x = \sum_{i=0}^{n} \alpha_i t^i \quad \alpha_i \in \mathbb{R}$

$\frac{dx}{dt} = \sum_{i=1}^{n-1} \alpha_i t^i$ note $\frac{d}{dt} t^n = n t^{n-1}$, hence

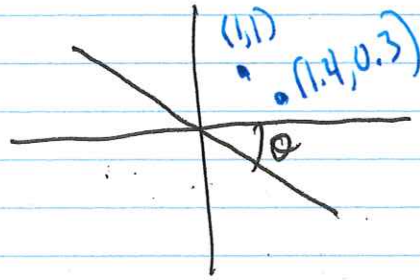
$\frac{d}{dt} 1 = 0 \quad \frac{d}{dt} t = 1 \quad \frac{d}{dt} t^2 = 2t \quad \frac{d}{dt} t^3 = 3t^2$

Take $n = 3$ so $x = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is in 4-dim space and $\frac{dx}{dt}$ lives in 3-dim space: $a_1 t^0 + 2a_2 t^1 + 3a_3 t^2$

so $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$

hence $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ maps $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$ to $\begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{pmatrix}$

(6) Find matrix of the reflection through the line $y = -2x/3$ in $\mathbb{R}^2$. Perform all multiplications



when $x = 1$ $y = -2/3$

$\tan\theta = -\frac{2}{3} \Rightarrow \theta = \arctan\left(-\frac{2}{3}\right)$

$\theta \approx -33.7$

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 0.83 & 0.55 \\ -0.55 & 0.83 \end{pmatrix}$$

For example, the point $z = (1,1)$ transforms to

$$z' = Tz = \begin{pmatrix} 0.83 & 0.55 \\ -0.55 & 0.83 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 1.4 \\ 0.3 \end{pmatrix}$$

(7) Prove that the trace $(AB) = \text{trace}(BA)$, both $A, B \in \mathbb{R}^{n \times n}$

$\text{trace } A = \sum_{i=1}^{n} a_{ii}$ let $C = AB$, $C' = BA$

$\cancel{\text{trace}(AB) = \sum_{i=1}^{n} a_{ij}}$ $\text{trace}(C) = \text{trace}\left(\sum_{l=1}^{n} a_{jl} b_{lk}\right)$

$= \text{trace}(c_{jk}) = \sum_{j=1}^{n} c_{jj}$

$\therefore \text{trace } C = \sum_{j=1}^{n}\left(\sum_{l=1}^{n} a_{jl} b_{lj}\right)$

$\text{trace } C' = \sum_{j=1}^{n}\left(\sum_{l=1}^{n} b_{jl} a_{lj}\right) = \sum_{l=1}^{n}\left(\sum_{j=1}^{n} b_{lj} a_{jl}\right) = \sum_{l=1}^{n}\left(\sum_{j=1}^{n} a_{jl} b_{lj}\right) = \text{Trace}(C)$

(8) Let X and Y be subspaces of V. Prove that $U \cap V$ is a subspace of V. Let $V = \{v_i\}_{i=1}^{p}$

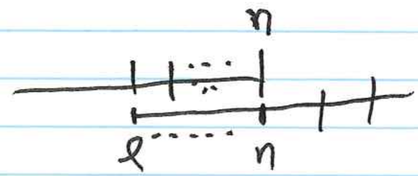
case(i) let $v_1 = \phi$, and $X = v_1$, $Y = \sum_{i=2} v_i$

$\Rightarrow X \cap Y = V$. Same for $X = \sum_{i=1}^{p} v_i$ and $Y = \phi$

case(ii) let $X = \sum_{i=1}^{n} v_i$ and $Y = \sum_{i=n+1}^{p} v_i$, then $V = X \cap Y$, trivially. here $n \le 1 \dots \le p$.

case(iii) let $X = \sum_{i=1}^{n} v_i$ $Y = \sum_{i=n+1}^{l} v_i$ then $X \cap Y \in V$ $1 \le n \le p$, $l > n$ and $l \le p$.

case(iv) let $X = \sum_{i=1}^{n} v_i$ $Y = \sum_{i=l}^{p} v_i$

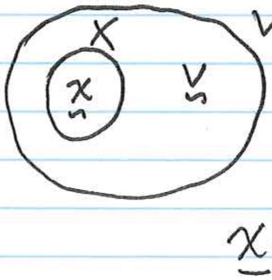


where $n \le p$, $l \le p$ & $l \ge 1$

then $X \cap V = \left(\sum_{i=1}^{l-1} v_i + \sum_{i=l}^{n} v_i\right) \cap \left(\sum_{i=n+1}^{p} v_i + \sum_{i=l}^{n} v_i\right) = \sum_{i=l}^{n} v_i$

$\sum_{i=l}^{n} v_i \in V$

(9) Let X be a subspace of V, let $\underset{\sim}{v} \in V$, $\underset{\sim}{v} \notin X$. Prove that if $\underset{\sim}{x} \in X$ then $\underline{x} + \underline{v} \notin X$



Since $\underset{\sim}{v}$ cannot be written entirely as a linear combination of vectors in X then $\underline{x} + \underline{v}$ cannot be written entirely as a linear combination of vectors in $X \Rightarrow x + v \notin X$.

(10) Show that $T: p(x) = \int_{-\infty}^{x} f(s)\,ds \mid f(x), p(x) \in C^{\infty}(\mathbb{R})$

is a linear operator

(a) Additivity:

$$p(x) = \int_{-\infty}^{x_1} f(s)\,ds + \int_{x_1}^{x} f(s)\,ds \quad , \text{where } x_1 \le x$$

(b) Multiplication by a scalar

$$\alpha p(x) = \int_{-\infty}^{x} \alpha f(s)\,ds = \alpha \int_{-\infty}^{x} f(s)\,ds$$

(c) $0 = \int_{-\infty}^{x} f(s)\,ds + \int_{-\infty}^{x} g(s)\,ds \quad \text{if } g(x) = -f(x)$

(11) Determine whether the Transformation is linear or not

(a) $T\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_0 \end{pmatrix}$ No. The 0 vector is not present

(b) $T\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_0 - x_2 \\ x_1 \end{pmatrix}$ Yes. $\mathbb{R}^3 \to \mathbb{R}^2$

$$T = \begin{cases} a_1 x_0 + b_2 x_1 + c_3 x_2 = x_0 - x_2 \\ a_1 x_0 + b_2 x_1 + c_3 x_2 = x_1 \end{cases}$$

$$T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$