Homework 1

① Verify that each of the following is a vector space

(a) 2x2 matrices of the form

let $V = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$

clearly, closed by addition, for $c, d \in \mathbb{R}$

take $\underline{u} + \underline{v} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a+c & 0 \\ 0 & d+b \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \equiv W$

where $c_1, c_2 \in \mathbb{R}$, $W \in V$

Closed under scaler multiplication, take α a scalar $\in \mathbb{R}$

$(\alpha \notin \mathbb{C})$ $\alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} = \begin{pmatrix} c_3 & 0 \\ 0 & c_4 \end{pmatrix} \equiv z$

$c_3, c_4 \in \mathbb{R} \therefore z \in V$

it is easy to see how to define $\phi \in V$:

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} = \phi$ if $a = -e$, $b = -f$

clearly $e, f \in \mathbb{R}$.

(b) The collection of polynomials of the form

$U = \{ (a_0 + a_1 x + a_3 x^3) \mid a_i \in \mathbb{R} \quad i = 0, 1, 3 \}$

Addition: $a_0 + a_1 x + a_3 x^3 + b_0 + b_1 x + b_3 x^3 = c_0 + c_1 x + c_3 x^3 \equiv P_3$

c_0, c_1, c_3 are real, hence $P_3 \in U$

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scalar multiplication: if $\alpha \in \mathbb{R}$

$$\alpha(a_0 + a_1 x + a_3 x^3) = \alpha a_0 + \alpha a_1 x + \alpha a_3 x^3$$

$$= c_0 + c_1 x + c_3 x^3 \equiv \mathcal{P}_3 \quad c_i \in \mathbb{R}$$

hence $\mathcal{P}_3 \in U$

the zero ϕ vector simply requires $b_0 + b_1 x + b_3 x^3 = \underline{r}$

to have $b_0 = -a_0$, $b_1 = -a_1$, $b_3 = -a_3$ so that $r \in U$

2. Determine if each set is LI:

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Form $c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \phi$

$c_1 = -c_2$ and $2c_1 + c_2 = 0$ can only be satisfied if

$c_1 = c_2 = 0$ $\therefore$ LI

(b) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 11 \\ 7 \end{pmatrix} \right\}$

set $c_1 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 11 \\ 7 \end{pmatrix} = 0$

leads to $\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & 4 & 11 & 0 \\ 1 & 3 & 7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 7 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

with inf many solutions, i.e. more than the trivial solution

$\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \middle| c_3 \in \mathbb{R} \right\}$ $\therefore$ not LI

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(2) $\left\{ \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} \right\}$

clearly the second element is a linear combination of either the first or last element.

So not LI

(3) Prove/Disprove: $\exists$ a basis (p_0, p_1, p_2, p_3) of $P_3(\mathbb{R})$ st none of the polynomials p_0, p_1, p_2, p_3 are degree 2?

Take $p_0 = a_0 x^0$

$p_1 = a_1 x^1$

$p_2 = a_2 x^2 + a_3 x^3$

$p_3 = a_4 x^3$

$\Rightarrow \left\{ (x^0, x^1, x^2 + bx^3, x^3) \mid b \in \mathbb{R} \setminus \{0\} \right\}$ has LI bases, none of which is a polynomial nor of degree 2 (since $b \neq 0$)

(4) A basis for $\mathbb{R}^\infty$ is

$\begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, etc.

(5) Give 2 different basis for $\mathbb{R}^3$. Verify each is a basis:

Take $(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, clearly these are LI

and $\underset{\sim}{u} \in \mathbb{R}^3$ $\underset{\sim}{u} = \sum_{i=1}^{3} \alpha_i \underline{v}_i \in \mathbb{R}^3$

$\alpha_i \in \mathbb{R}$

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Take $\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$

To verify it spans $\mathbb{R}^3$, let $x, y, z \in \mathbb{R}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}1\\1\\1\end{pmatrix}$$

has a solution $c_1 = x - y$, $c_2 = y - z$, $c_3 = z$

also $\begin{pmatrix}0\\0\\0\end{pmatrix} = c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + c_3 \begin{pmatrix}1\\1\\1\end{pmatrix}$ requires $c_i = 0$ $i = 1, 2, 3$.

(6) Represent the vector $\begin{pmatrix}3\\-1\end{pmatrix} = \underset{\sim}{v}$ with respect to each of these bases:

$$B_1 = \left\{ \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\} \qquad B_2 = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \right\}$$

Need to find c_1, c_2 s.t.

for B_1 $\quad \underset{\sim}{v} = c_1 \begin{pmatrix}1\\-1\end{pmatrix} + c_2 \begin{pmatrix}1\\1\end{pmatrix}$ $\quad c_1 = 2, \ c_2 = 1$

for B_2 $\quad \underset{\sim}{v} = c_1 \begin{pmatrix}1\\2\end{pmatrix} + c_2 \begin{pmatrix}1\\3\end{pmatrix}$

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(7) Give a basis for the span of each set:

(a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 \\ 6 \end{pmatrix} \right\}$. Set up

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 12 \\ 3 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & -6 \end{pmatrix}$$

So a basis for the span is $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -6 \end{pmatrix}$

(b) $\{ (x+x^2), 2-2x, 7, 4+3x+2x^2 \}$

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & -2 & 0 \\ 7 & 0 & 0 \\ 4 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

So a basis is $(2-2x, x+x^2, -5x^2)$