ECI

MARKOV CHAIN (By example)

Consider a state vector that changes in time, $x(t_n)$ where time $t_n = t_0, t_1, t_2 \cdots$ are subsequently later times (from Macdonald & Ridge, p202 (1988) sociology study).

$$x(t_n) \equiv x_n = \begin{pmatrix} x_U \\ x_M \\ x_L \end{pmatrix}(n)$$

X_n represents the distribution of work occupations at time t_n.

where $x_U \equiv$ # of upperlevel (executives & professionals)

$x_M \equiv$ # of mid level (managers & skilled workers)

$x_L \equiv$ # of lower level (unskilled workers)

The sociologists wanted to study the mobility of workers across these distributions, as a function of time. The following table is generated from data:

$$\begin{array}{c} \\ U \\ M \\ L \end{array} \begin{array}{c} \begin{array}{ccc} U & M & L \end{array} \\ \begin{pmatrix} 0.60 & 0.29 & 0.16 \\ 0.26 & 0.37 & 0.27 \\ 0.14 & 0.34 & 0.57 \end{pmatrix} \end{array} \equiv T = T_{ij}$$

This is the transition (probability) matrix T. Because these are probabilities, all entries are between 0 and 1. The column-wise sum $\sum_{i=1}^{3} T_{ij} = 1$ for columns $j = 1, 2, 3$.

The entry 0.6 indicates that a child of a U has a 60% of being U. The entry 0.29 indicates that probability of a child of an M becoming a U, the entry 0.16 indicates probability that a child of an L becomes a U, etc.

EC2

So x_{n+1} is the children population, x_n is the parent population, hence

$$x_{n+1} = T x_n \text{ , and}$$

$$x_{n+2} = T x_{n+1} \text{ , etc}$$

~~so~~ so $X_n = T^n X_0$

X_0 is the first generation, x_n the n^{th} generation..

$X_0 = \begin{pmatrix} 0.12 \\ 0.32 \\ 0.56 \end{pmatrix}$ is the percentage distribution of the first generation.

x_0 was estimated by the sociologists, they also used data to determine the transition matrix T. They (must have) found that T does not change appreciably over time.

The states $x_0, x_1, x_2 \ldots$ are components of the MARKOV chain.

Because all you need to find x_{n+1} is ~~that x_n~~ is x_n, we call the chain "Markovian". If it were the case that x_{n+1} requires $\{x_{n-\ell}\}_{\ell \geq 0}$

i.e. requires knowledge of several prior states we then call the chain NON MARKOVIAN.

So all we need to find x_{n+1} is x_n and T:

$$x_{n+1} = T x_n .$$

We might as, after m generations, how does the present 0 generation will distribute itself?

We solve $x_m = T^m x_0$

We might ask, what happens after many many generations?

We solve $\lim_{m\to\infty} x_m = \lim_{m\to\infty} T^m x_0$

(1) Show that, if $x_0 = \begin{pmatrix} 0.12 \\ 0.32 \\ 0.56 \end{pmatrix}$, after 5 generations $x_5 = \begin{pmatrix} 0.33 \\ 0.33 \\ 0.34 \end{pmatrix}$

(2) Show that, if $x_0 = \begin{pmatrix} 0.12 \\ 0.32 \\ 0.56 \end{pmatrix}$, $x_\infty \equiv \lim_{m\to\infty} x_m = \begin{pmatrix} 1.0 \\ 0 \\ 0 \end{pmatrix}$

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The Leontief ~~(closed)~~ Input-output Model

Purchased from	Inputs consumed per unit of output		
	Manufacturing	Agriculture	Services
Manufacturing	0.5	0.4	0.2
Agriculture	0.2	0.3	0.1
Services	0.1	0.1	0.3

The "consumption" matrix $A = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}$

Let $p \equiv \begin{pmatrix} p_M \\ p_A \\ p_S \end{pmatrix}$ represent the (input) in dollars of production resources of type p_M (manufacturing) p_A (agriculture) and p_S (services).

Let y be the (target) demand $y = \begin{pmatrix} y_M \\ y_A \\ y_S \end{pmatrix}$

A question might be: can a certain demand y be met, given that we start with p resources and it takes Ap to produce y?

find p such that $p - Ap = y$

or $p = (I-A)^{-1} y$

If demand y and production p are to be nonnegative we require that

$(I-A)^{-1}$ be a non-negative matrix.

If A is "bigger" than I then production is too high and then nothing's left as output.

(3) What production level p can satisfy the demand \$50 manufacturing, \$30 agriculture, \$20 services?

(4) Use the spectrum of A to prove that $I-A$ is invertible, and in addition, a way to show that $(I-A)^{-1}$ is non-negative.

The geometric series: consider ~~|x|<1~~ x, a number other than $|x|=1$: we can sum

$$1+x+x^2+\cdots x^m = \sum_{i=0}^{m} x^i$$

Multiply $\sum_{i=0}^{m} x^i$ by $(1-x)$ to show that

$$\sum_{i=0}^{m} x^i = \frac{1-x^{m+1}}{1-x}.$$

(a) Compute $\sum_{i=0}^{100} (0.1)^i$.

If $|x|<1$ the infinite series $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

(b) Compute $\sum_{i=0}^{\infty} (0.1)^i$.

(c) Assume $A = SDS^{-1}$ and the spectrum of A does not contain eigenvalues with modulus 1.

Then

$$\sum_{i=0}^{m} A^i = (1-A)^{-1}(1-A^{m+1})$$

(d) Assume $A = SDS^{-1}$ and $\sigma(A)$ contains eigenvalues that are <u>all</u> smaller than 1 in modulus.

$$(I-A)^{-1} \approx I + A + A^2 + \cdots + A^m$$

and $$(I-A)^{-1} = \sum_{i=0}^{\infty} A^i$$