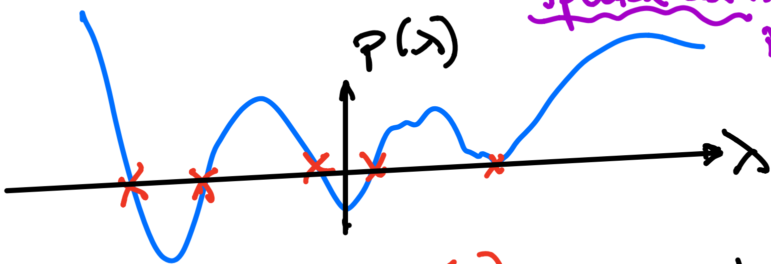
Spectrum of finite dimensional linear transformations: if we know how to compute the spectrum of matrices, we, in principle know to compute spectrum of other finite dimensional linear transformations, i.e. there's an isomorphism between the matrix problem and the finite dimensional operator.

The eigenvalue problem: find $\underset{\sim}{v} \in \mathrm{Null}(Ax - \lambda I)$

i.e. $(A - \lambda I)\underset{\sim}{v} = 0$

To find λ: Solve $p(\lambda) = 0$

characteristic equation



Special case: roots are real in this caricature. In general, roots are complex.

$\lambda \in \sigma(A)$ are $\times$, but note that λ can be complex, generally.

$$p(\lambda) = \det(A - \lambda I) = 0$$

$p(\lambda) = 0$ turns out to come from a solvability condition of $(A - \lambda I)v = 0$.

If $v \neq 0$ then $A - \lambda I$ is, in some sense, 0.

For $A \in \mathbb{F}^{n \times n}$ with n eigenvalues

① $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$ (counting multiplicities)

② $\det(A) = \prod_{i=1}^{n} \lambda_i$ (counting multiplicities)

Eigenvalues of a triangular matrix

$$A = \begin{pmatrix} a_{11} & & X \\ & a_{22} & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

We say A is upper triangular matrix. Require a_{ii} not all zero.

$$B = \begin{pmatrix} b_{11} & & 0 \\ & b_{22} & \\ & \ddots & \\ X & & b_{nn} \end{pmatrix}$$

We say B is lower triangular matrix. Require b_{ii} not all zero.

$$\boxed{\begin{gathered} \det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0 \\ \det(B-\lambda I) = (b_{11}-\lambda)(b_{22}-\lambda)\cdots(b_{nn}-\lambda) = 0 \\ \Rightarrow \text{So for } n\times n \text{ triangular matrices, the eigenvalues are the diagonal entries.} \end{gathered}}$$

ex) $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 3 & 1 & 1 \end{pmatrix}$ determine the eigenvalues

4, 1 are single e'values

0 is e'value of multiplicity 2.

An $n\times n$ matrix does not have an inverse if 0 is one of its eigenvalues

$tr(A) = 4+0+0+1 = 5$; $\det(A) = 4\cdot 0\cdot 0\cdot 1 = 0$

Eigenvalues and Invertibility:

Let $A \in \mathbb{F}^{n \times n}$. A^{-1} exists iff $0 \notin \sigma(A)$.

Can see this is the case; take $\lambda = 0$, then

$$Ax = 0Ix, \text{ equivalent to } Ax = 0.$$

But $Ax = 0$ has a non-trivial solution iff A is NOT invertible.

(Recall $Ax = 0$ has a trivial solution if the columns of A form an LI set.)

The Invertibility Theorem: Suppose $A \in \mathbb{F}^{n \times n}$, the following are ALL FALSE or ALL TRUE

(a) $\exists A^{-1}$

(b) A is the product of elementary matrices (matrices that perform row ops to deliver row echelon form)

(c) A is "row equivalent" to the identity matrix (i.e. it has n pivots)

(d) $Ax=0$ has only $x=0$ as a solution,

(e) A^T is invertible

(f) $Ax=b$ has at least 1 solution for each $b \in \mathbb{F}^n$

(g) $\delta(A) \neq 0$.

If $Mv=0$ has non-trivial solutions $(v \neq 0)$ M has to be "zero" in some sense. To get an understanding of why this is so, let's consider $Mx=b$, where $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. We want to find $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

By Cramer's Rule

(A) $x_1 = \dfrac{\det \begin{bmatrix} b_1 & \beta \\ b_2 & \delta \end{bmatrix}}{\det M}$ $\quad x_2 = \dfrac{\det \begin{bmatrix} \alpha & b_1 \\ \gamma & b_2 \end{bmatrix}}{\det M}$. (B)

So $\det M \neq 0$ is a condition for solvability. Consider $Mx = 0$.

Recast (A) $\det M \, x_1 = \det \begin{bmatrix} 0 & \beta \\ 0 & \delta \end{bmatrix} = 0$

Recast (B) $\det M \, x_2 = \det \begin{bmatrix} \alpha & 0 \\ \gamma & 0 \end{bmatrix} = 0$

So for x_1 & x_2 not both be zero we require $\det M = 0$. Now, suppose $M = A - \lambda I$. So $(A - \lambda I)v = 0$ has $v \neq 0$ if $\det(A - \lambda I) = 0$ //

Cramer's rule applies to $Ax = b$, where A is $n \times n$ matrix (See Lax "Linear Algebra").

$x = \dfrac{\mathrm{adj}(A)}{\det(A)} b = A^{-1} b$. The adjunct matrix adj A is

$$\mathrm{adj}\,A = \{b_{ij}\} = (-1)^{i+j} \det \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1d} \\ \vdots & & & & & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,d} \\ a_{i+1,1} & \cdots & & & \cdots & a_{i+1,d} \\ \vdots & & & & & \\ a_{d,1} & \cdots & & & \cdots & a_{d,d} \end{pmatrix}.$$

$i, j = 1, 2, \ldots, d$.

You don't have to memorize/interpret this formula. It is not practical.

Cramer's is useful theoretically, but it is not used to solve large systems of equations computationally. It is very numerically sensitive to errors due to finite precision. //

A useful fact, demonstrated on the 2x2 case. Spse
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $p(\lambda) = 0$ is

$$\lambda^2 - (a+d)\lambda + ad - cb = 0, \text{ or}$$

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

Roots of $p(\lambda) = 0$ are then

$$\lambda_{1,2} = \frac{1}{2} Tr(A) \pm \frac{1}{2}\sqrt{[Tr A]^2 - 4 Det(A)}$$

//

An important application of Spectral Theory

Diagonalization of an operator (e.g. of a matrix). Caution: not all linear transformations, be they finite or infinite dimensional, are diagonalizable.

For $A \in \mathbb{F}^{n \times n}$, we want to find S, D s.t.

$$A = SDS^{-1} \quad \text{or} \quad AS = SD$$

D is a diagonal matrix

S has to be invertible

//

Note: $\det(A) = \det(SDS^{-1})$

$= \det(S)\det(D)\det(S^{-1})$, but

$\det S = \dfrac{1}{\det(S^{-1})}$.

$\therefore \det(A) = \det(D) \Rightarrow A \,\&\, D$ are similar matrices.

if $A = SDS^{-1}$

then D has to be related to λI:

$$A - \lambda I = SDS^{-1} - \lambda I = SDS^{-1} - \lambda SS^{-1} = S(D-\lambda)S^{-1}$$

so $A - \lambda I$ is similar to $D - \lambda I$

$$\therefore \det(A - \lambda I) = \det(D - \lambda I)$$

Characteristic polynomials of similar matrices coincide.

So $\sigma(A)$ can be used to construct D.

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

//