Invariant Subspaces

Section 3B LADR

We denote by $T \in \mathcal{L}(V,W)$ $\quad T: V \to W$

T a linear transformation, V & W vector spaces.

We denote by $P \in \mathcal{L}(V)$, $P: V \to V$, P the linear transformation.

For $T \in \mathcal{L}(V,W)$: The kernel of T or the Null Space of T: null(T) is the subset of V consisting of vectors that T maps to 0

$$\text{null}(T) = \{v \in V : Tv = 0\}$$

The null(T) is a subspace of V.

Range of T: range(T) is a subset of vectors in W consisting of vectors of the form Tv, for $v \in V$

$$\text{range}(T) = \{Tv : v \in V\}$$

The range(T) is a subspace of W. //

For $T \in \mathcal{L}(V,W)$ and for V is finite-dimensional, the range of T is finite dimensional.

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

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Examples of $\mathcal{L}(V,W)$:

ex) $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})) = \mathcal{L}(\mathcal{P}(\mathbb{R}))$

$\mathcal{P}(\mathbb{R})$ are polynomials of any degree

define $Dp = \frac{d}{dx} p$

where $p \in \mathcal{P}(\mathbb{R})$

ex) let $Tp = \int_0^1 p \, dx$ $\quad$ T is a linear operator

$T \in \mathcal{L}(\mathcal{P}, \mathbb{R})$

. //

Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$

then $S+T$ and λT are linear maps

from V to W such that

$$(S+T)v = Sv + Tv$$

$$\lambda T v = \lambda (Tv)$$

$$\forall v \in V$$

hence, $\mathcal{L}(V,W)$ is itself a vector space! ✓

Invariant Subspace: S'pse $T \in \mathcal{L}(V)$.

A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in V$ //

ex) S'pse $T \in \mathcal{L}(V)$. Show that each of the following are subspaces of V (invariant under T):

(a) $\{0\}$. If $u \in \{0\}$ then $u = 0 \Rightarrow$

$Tu = 0 \in \{0\}$

$\{0\}$ is invariant under T

(b) If $u \in V$ and $Tu \in V \Rightarrow$
"V is invariant under T"

(c) null(v): if $u \in$ null(v) then $Tu = 0 \in$ null(T)
"null(T) is invariant under T"

Spectral Theory

Chapter 5 LADR (5A, 5C)

(Eigenvalues & Eigenfunctions)

We learn to construct the simplest possible invariant subspaces. Their simplicity make them extremely useful!

Subspaces of dimension 1:

Take $v \in V$ with $v \neq 0$. Let U equal the set of scalar multiples of v:

$$U = \{ \lambda v : \lambda \in \mathbb{C} \} = \text{span}(v),$$

U is a 1-dimensional subspace of V. If U is invariant under an operator $T \in \mathcal{L}(V)$.

$\Longrightarrow Tv \in U$

and hence $\exists$ a scalar $\lambda \in \mathbb{C}$ st

$$Tv = \lambda v$$

then span(v) is a 1 dim subspace of V is invariant under T.

Notation: $T|_U$ "restricted to". T is restricted to subspace U.

So the idea is to effect the decomposition

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$$

$T|_{U_j}$ maps U_j onto itself.

Eigenvalue: Spse $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\exists$ $v \in V$ st $v \neq 0$ s.t. $Tv = \lambda v$

Eigenvector: Spse $T \in \mathcal{L}(W)$ and $\lambda \in \mathbb{C}$ an eigenvalue of T. A vector $v \in V$ is the eigenvector of T corresponding to λ if $v \neq 0$ $Tv = \lambda v$.

$$\boxed{\text{so } v \in \operatorname{Null}(T - \lambda I)}$$

ex) Spse $T \in \mathcal{L}(\mathbb{F}^2)$,

$$T\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

so T is a linear transformation that rotates by $90°$.

(a) Restrict $\mathbb{F}$ to $\mathbb{R}$:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

is there a λ s.t.

$$\left.\begin{aligned} -z &= \lambda w \\ w &= \lambda z \end{aligned}\right\} \quad -z = \lambda^2 z$$

can λ be real? no.

(b) Allow $\mathbb{F}$ to be $\mathbb{C}$:

Ⓐ $-z = \lambda w$ $\quad w = \lambda z$ Ⓑ, combine Ⓐ & Ⓑ:

to get $-z = \lambda^2 z \Rightarrow (\lambda^2 + 1) z = 0$ if $z \neq 0$

$\lambda^2 + 1 = 0$. $\lambda_{1,2} = \pm i$ there are 2 constants that do the job.

for $\lambda = i$ $\quad \begin{pmatrix} w \\ -iw \end{pmatrix}$ $\qquad w \in \mathbb{C} \quad w \neq 0$

for $\lambda = -i$ $\quad \begin{pmatrix} w \\ +iw \end{pmatrix}$

eigenvalues $\qquad$ eigenvectors

T 2x2 matrix performs a rotation by 90° has 2 subspaces each spanned by a complex eigenvector

Thm: Let $T \in \mathcal{L}(V)$, S'pose $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, v_2, \ldots, v_m$ are the corresponding eigenvectors. Then $\{v_i\}_{i=1}^m$ are LI.

Pf: S'pse $\{v_i\}_{i=1}^m$ are LD. Let k be the smallest possible index s.t.

$$v_k \in \text{span}(v_1, \ldots, v_{k-1})$$

$\therefore \exists \{a_i\}_{i=1}^k \in \mathbb{F}$ st

$$v_k = \sum_{i=1}^{k-1} a_i v_i \quad (\maltese)$$

Apply T to $(\maltese)$ to get $Tv_k = T \sum_{i=1}^{k-1} a_i v_i$. That is,

$$\lambda_k v_k = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 \ldots a_{k-1} \lambda_{k-1} v_{k-1}.$$

Subtract $\lambda_k v_k = \lambda_k a_1 v_1 + \ldots + \lambda_k a_{k-1} v_{k-1}$

to get $0 = a_1(\lambda_k - \lambda_1) v_1 + a_2(\lambda_k - \lambda_2) v_2 \ldots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1}.$

but $\{v_i\}_{i=1}^{k-1}$ are LI $\therefore$ $a_i = 0$.

$\Rightarrow v_k = 0$ (since $v_k = \sum_{i=1}^{k-1} a_i v_i$)

This contradicts hypothesis that $\{v_i\}_{i=1}^{k}$ are LI //

Spectrum of T $\sigma(T(V))$ is the set of all eigenvalues of $T \in \mathcal{L}(V)$.

Finite Dimensional Problems, the matrix transformation:

The $\sigma(A)$ and the eigenvectors of

$A \in \mathbb{F}^{n \times n}$

$A: V \to V$ $\quad V$ is $\mathbb{F}^{n}$.

thes eigenvectors $v \in \text{Null}(A - \lambda I)$

I is the $n \times n$ identity matrix and $\lambda \in \mathbb{C}$

i.e. $(A - \lambda I)v = 0$.

To find $\lambda \in \sigma(A)$ we compute roots of the characteristic polynomial $p(\lambda)$.

Since $A \in \mathbb{F}^{n \times n}$ and want

$(A - \lambda I)v = 0$ with $v \neq 0$, then

$(A - \lambda I)$ must be "zero" in some sense.

$$\boxed{p(\lambda) = \det(A - \lambda I) = 0}$$

It is a polynomial of degree n in λ

There are n roots, possibly complex, some (or all) can be repeated.

Roots are eigenvalues of A, i.e.

$$\lambda \in \sigma(A)$$

if λ is not repeated we say it is a simple root

otherwise, a root of multiplicity $m \leq n$:

$$p(\lambda) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_k)^m(\lambda-\lambda_{k+1})\cdots$$

Finding eigenvalues & eigenvectors of matrix:

ex) $T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}$ find $\sigma(T)$ and eigenvectors of T

Form $(T-\lambda I) = 0$ I is 3×3 identity matrix

$$p(\lambda) = \det(T-\lambda I) = 0$$

is the polynomial with complex roots called the eigenvalues:

$$p(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda-2)^2 = 0$$

$\lambda_1 = 0$, is a simple root

$\lambda_{2,3} = 2$, root of multiplicity 2.

$(\lambda_1, \lambda_2, \lambda_3) \in \sigma(T)$

Find eigenvectors: $(A - \lambda_i I)v_i = 0 \quad i = 1, 2, 3.$

v_i are eigenvectors

for $\lambda_1 = 0$

$$\begin{pmatrix} 1-0 & 2 & 1 \\ 2 & 6-0 & -2 \\ 1 & 2 & 3-0 \end{pmatrix} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{v_1} = 0 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = v_1$$

for $\lambda_2 = \lambda_3 = 2$

$$\begin{pmatrix} 1-2 & 2 & 1 \\ 2 & 6-2 & -2 \\ -1 & 1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = v_2$$

and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = v_3$

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