The 4 Spaces associated with a Matrix (linear) Transformation

The linear transformation via a matrix is

$$Ax = b.$$

3 problems arise:

(i) Easiest given A and x, find b.
(ii) Harder given A and b, find x.
(iii) Hardest given x and b, find A

For (ii), the general solution is

$x = x_h + x_p$, where x_h (which can be nonzero) satisfies $Ax_h = 0$, and x_p solves $Ax_p = b$.

How can x_h be non-zero? our intuition is based upon a very peculiar problem: $ax - b = 0$, where a, x, b are scalars. For this case $x_h = 0$ always.

Let's remind ourselves of the mechanics of finding the 4 subspaces associated with a matrix transformation

$$A: \mathbb{R}^n \to \mathbb{R}^m :$$

ex) $A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix}$ $A \in \mathbb{R}^{m \times n}$

$m = 4$

$n = 6$

in matlab rref(A) (row operations to get the "echelon" form)

$B = \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ pivots

(a) Find column space of A

$$\mathrm{Col}(A) = \left(a_1, a_3, a_6 \right) = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 15 \\ 18 \end{pmatrix} \right)$$

$\dim(\mathrm{Col}(A)) = \text{rank} \equiv r = 3$.

Null Space of A: set up $Ax = 0$ $(\dagger)$

$\dim(\mathrm{Null}(A)) = n - r = 6 - 3 = 3$

$x = (x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6)^T$

Free parameters x_2, x_4, x_5

Referring to $(\dagger)$

$$\begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_2 = x_2 \\ x_3 = -2x_4 \\ x_4 = x_4 \\ x_5 = x_5 \\ x_6 = 0 \end{cases}$$

$$x_h = N_1 x_2 + N_2 x_4 + N_3 x_5$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{N_1} + x_4 \underbrace{\begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{N_2} + x_5 \underbrace{\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{N_3}$$

$Ax = 0$ means, find an x that is orthogonal to all the rows of A!

$$\therefore \; Null(A) = (N_1, N_2, N_3)$$

row space of A

$$row(A) = col(A^T)$$

$$A^T = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & 0 & 6 \\ -2 & -5 & 5 & 0 \\ 0 & -2 & 10 & 8 \\ 2 & 4 & 0 & 4 \\ 0 & -3 & 15 & 18 \end{pmatrix}$$

Could use $rref(A')$. The vectors spanning $row(A)$: the pivot rows of B:

$$row(A) = \left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$dim(row(A)) = r = 3$$

Find $\text{null}(A^T)$

$\dim[\text{null}(A^T)] = m - r = 4 - 3 = 1$

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ \multicolumn{4}{c}{\boxed{0 \quad {\scriptstyle 3\times 4}}} \end{pmatrix}$$

$A^T y = 0$ find y that is orthogonal to rows of A^T

$y_1 = -10 y_3$

$y_2 = 5 y_3$

$y_3 = y_3$

$y_4 = 0$

$$y = \begin{pmatrix} -10 \\ 5 \\ 1 \\ 0 \end{pmatrix} y_3$$

$$\text{null}(A^T) = \left(\begin{pmatrix} -10 \\ 5 \\ 1 \\ 0 \end{pmatrix} \right)$$

ex) $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \in \mathbb{R}^{m \times n}$

$m = 4$

$n = 5$

$A = \begin{bmatrix} | & | & | & | & | \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ | & | & | & | & | \end{bmatrix}$ $\quad a_i \in \mathbb{R}^m$

$i = 1, 2, .., n$

// $Ax = b$

$A : \mathbb{R}^n \to \mathbb{R}^m$; $x \in \mathbb{R}^n = \mathbb{R}^5$

$b \in \mathbb{R}^m = \mathbb{R}^4$

Find Col(A). The row echelon form of A:

$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix}$

Note: $b_2 = 4b_1 \quad b_4 = 2b_1 - b_3$

$\dim(\mathrm{Col}(A)) = \text{rank} = r = 3$

$$\mathrm{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 2 \\ 8 \end{pmatrix} \right\}$$

$$\mathrm{Col}(A) = \mathrm{span}(a_1, a_3, a_5)$$

$(*)$ $Ax = 0$ is null space problem

(Note: $x = \emptyset$ is always a solution of $(*)$)

$$\dim(\mathrm{Null}(A)) = n - r = 5 - 3 = 2$$

$(*)$
$$\begin{pmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_2(A) \\ \vdots \\ \mathrm{row}_n(A) \end{pmatrix} \underline{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

means that every row of A is $\perp$ (orthogonal) to x.

$$x = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_4$$

$$\mathrm{Null}(A) = \left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\text{row}(A) = \text{col}(A^T)$

$$A^T = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 4 & 12 & 8 & 20 \\ 0 & 1 & 1 & 2 \\ 2 & 5 & 3 & 8 \\ -1 & 5 & 2 & 8 \end{pmatrix}$$

To find vectors of row(A), go back to B

$\dim(\text{row}(A)) = r = 3$

$$\text{row}(A) = \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\text{Null}(A^T)$ $\qquad \dim(\text{Null}(A^T)) = m - r = 4 - 3 = 1$

$A^T y = 0$

$$y = \begin{pmatrix} -1/3 \\ 12/3 \\ 1 \\ -11/3 \end{pmatrix} y_4 \qquad \text{Null}(A^T) = \left\{ \begin{pmatrix} -1/3 \\ 12/3 \\ 1 \\ -11/3 \end{pmatrix} \right\}$$

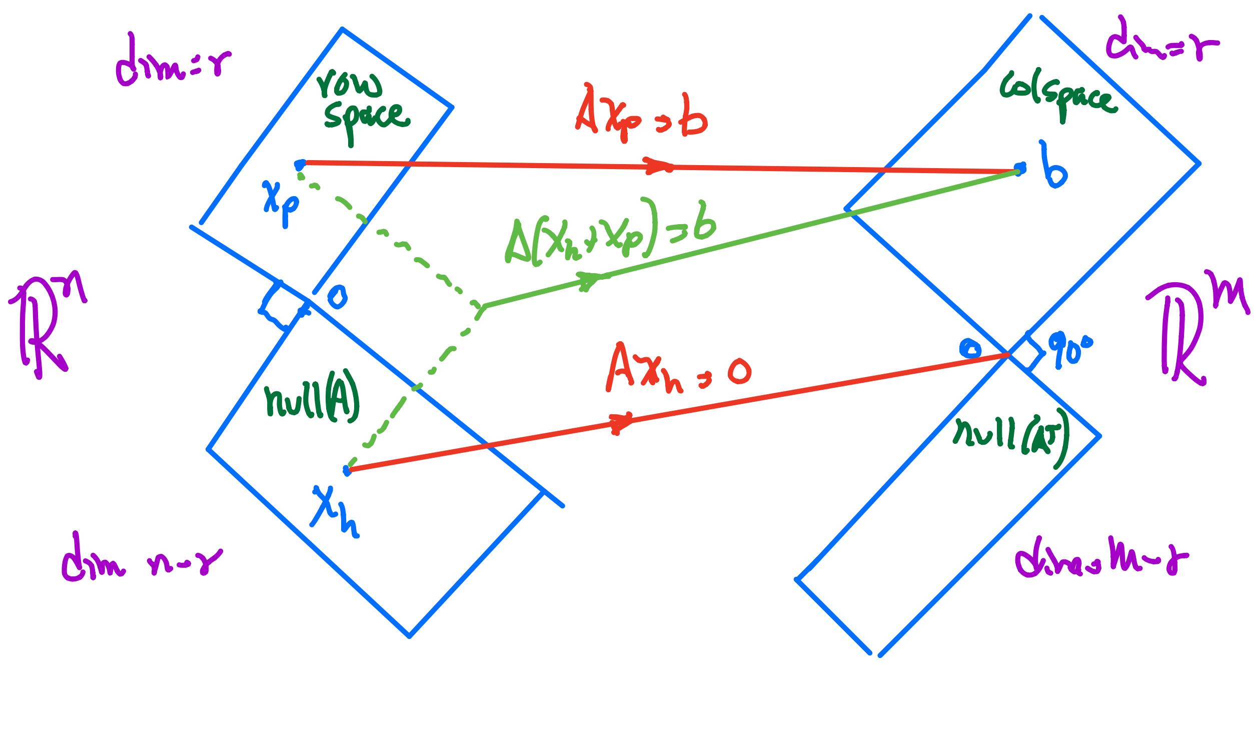
There are 4 fundamental Subspaces of a Matrix Transformation

$(*)\ Ax = b \qquad A \in \mathbb{R}^{m \times n} : \mathbb{R}^n \to \mathbb{R}^m.$

$x = x_n + x_p$

$(*)$ says that $a_1 x_1 + a_2 x_2 \cdots a_n x_n = b$; $x_i = (x_i^h + x_i^p)$ $i = 1, 2, .., n.$

where $A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}.$



(See Gil Strang's Youtube lectures on this decomposition)

Do these by inspection:

ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 2} \longrightarrow B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\text{col}(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ rank $= 1$

$\text{Null}(A) : \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\dim(\text{Null}(A)) = n - r = 1$

Row Space : (any multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$)

$\text{Null}(A^T)$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$A^T y = 0$ $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Try $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ this one on your own